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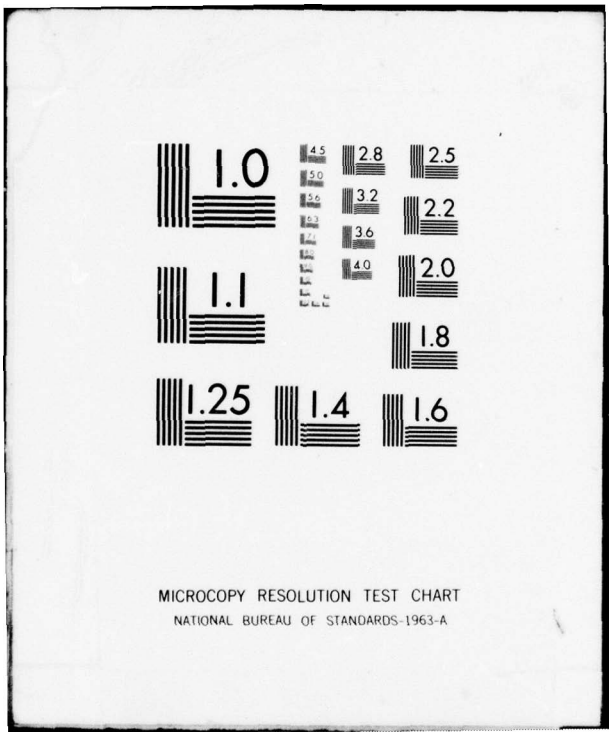
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(6) AN EXACT TEST FOR THE SEQUENTIAL ANALYSIS OF VARIANCE*

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CHAPTER 1

SEQUENTIAL FIXED EFFECTS ONE-WAY ANALYSIS OF VARIANCE

1.0 INTRODUCTION

This first chapter of the thesis will consider both the fixed and sequential analysis of variance tests. For the fixed sample test the discussions consist of the statistical model, the optimum properties of the test, and the operating characteristic (OC) function. Each of these concepts is important for the consideration of the sequential analysis of variance test.

The sequential analysis of variance test (termed SANOVA) is first discussed from a historical perspective. Further discussions consist of the experimental procedure, the test statistic, and the test statistic decision rule or regions. The OC and average sample number (ASN) functions are also defined. These functions are extremely helpful for designing SANOVA tests.

1.1 ONE-WAY FIXED EFFECTS ANALYSIS OF VARIANCE

Analysis of variance, a term introduced into statistics by R.A. Fisher (1918, 1925, 1935), is a statistical technique for analyzing measurements depending upon several kinds of effects operating simultaneously. In general, this technique consists of a body of tests of hypotheses, methods of estimation, etc., using statistics which are linear combinations of sums of squares of linear functions of the observed measurements. The simplest case in which analysis of variance is applied, is the one-way classification, in which the observations depend upon only one factor.

In the one-way layout, a population is stratified into m subpopulations according to some characteristic or factor and n_i independent observations are taken from each of k of the m subpopulations ($i = 1,\ldots,k$). Let the jth observation from population i be denoted by x_{ij} where $i = 1,\ldots,k$ and $j = 1,\ldots,n_i$. Given that population i has mean $\mu+\sigma_i$ and standard deviation σ_i, the statistical model employed in the one-way layout is

$$x_{ij} = \mu + \sigma + e_{ij} , \quad i = 1,\cdots,k; \quad j = 1,\cdots,n_i$$

with the parameters $\delta_1,\ldots,\delta_k$ satisfying the following condition

$$n_1\delta_1+\ldots+n_k\delta_k = 0$$

The parameter δ_i is referred to as the differential effect due to the factor at level i.

The usual hypothesis of interest is whether $\delta_1 = \delta_2 = .. = \delta_k = 0$, which is equivalent to the hypothesis of the equality of the k means. The analysis of the effect of the factor depends upon whether $k < m$ or $k = m$. Eisenhardt (1947) was the first to differentiate between the two situations. He used the terms Model I or a fixed effects model as the case where the sample consists of all groups in the population, i.e., $k = m$, and Model II or a random effects model as the case where the interest is in the population from which the sample came, i.e., $k < m$. This thesis will be concerned with only fixed-effects one-way analysis of variance.

The analysis of variance technique requires several assumptions. Specifically, it is assumed that the observations from each of the subpopulations are random variables distributed normally with mean $\mu + \delta_i$ and standard deviation $\sigma = \sigma_i$ for all i. In other words, the model may be expressed as

$$x_{ij} = \mu + \delta_i + e_{ij} \qquad i = 1,\ldots,k;\ j = 1,\ldots,n$$

$$x_{ij} \sim N(\mu + \delta_i, \sigma)$$

$$e_{ij} \sim N(0, \sigma)$$

and

$$\text{cov}(x_{ij}, x_{\ell m}) = 0.$$

With this model the hypotheses

H_0: $\mu_1 = \mu_2 = \cdots \mu_k$ vs. H_1: not all means equal

can be tested with the following statistic

$$F_{cal} = \frac{\sum_{i=1}^{k} n_i (\bar{x}_i - \bar{\bar{x}})^2/(k-1)}{\sum_{i=1}^{k} \sum_{j=1}^{n_i} (x_{ij} - x_i)^2/(N-k)}$$

where

$$N = \sum_{i=1}^{k} n_i$$

$$\bar{x}_i = \frac{1}{n_i} \sum_{j=1}^{n_i} x_{ij}$$

$$\bar{\bar{x}} = \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} x_{ij}$$

This statistic can be shown (Kempthorne, 1952) to be distributed as a noncentral F variate with (k-1, N-k) degrees of freedom and noncentrality parameter $\bar{n}\lambda$, where

$$\lambda = \frac{\sum_{i=1}^{k} \delta_i^2 n_i}{\sigma^2} = \frac{\sum_{i=1}^{k} (\mu_i - \bar{\mu})^2 n_i}{\sigma^2} \quad \text{with} \quad \bar{\mu} = \frac{1}{k} \sum_{i=1}^{k} \mu_i$$

$$\text{and} \quad \bar{n} = \frac{1}{k} \sum_{j=1}^{k} n_i$$

The density function of a noncentral F variate with v_1, v_2 degrees of freedom and noncentrality parameter λ is given by:

$$f_{v_1,v_2,\lambda}(x) = \frac{e^{-\frac{1}{2}\lambda}\, v_1^{\frac{1}{2}v_1}\, v_2^{\frac{1}{2}v_2}\, x^{\frac{1}{2}v_1-1}}{B(\frac{1}{2}v_1,\ \frac{1}{2}v_2)\ (v_2+v_1x)^{\frac{1}{2}(v_1+v_2)}} \sum_{j=0}^{\infty} \left[\frac{\frac{1}{2}\lambda v_1 x}{v_2 + v_1 x}\right]^j \frac{\Gamma\left(\frac{1}{2}(2j + v_1 + v_2)\right)}{j!\,\Gamma(\frac{1}{2}v_2)\ \Gamma\frac{1}{2}(2j + v_1)}$$

(Johnson and Kotz, 1970). (1.1.1)

If the null hypothesis is true, the distribution of F_{cal} is a central F distribution with k-1, N-k degrees of freedom. Hence, if the hypothesis is rejected whenever F_{cal} is greater than the $100(1-\alpha)\%$ point of this distribution, that is

$$F_{cal} > F^*_{k-1,N-k,1-\alpha}$$

then the significance level of the test will be α.

The operating characteristic curve of the test, that is, the probability of accepting H_0 is given by $\Pr\{F_{cal} \leq F^*_{k-1,N-k,1-\alpha}\}$. Since $F_{cal} \sim F_{k-1,N-k,\bar{n}_\xi}$ the OC of the test is characterized by the parameter $\xi = \bar{n}\lambda$, i.e.

$$OC(\lambda) = \Pr\{F_{k-1,N-k,\xi} \leq F^*_{k-1,N-k,1-\alpha}\}$$

Several sets of tables and curves have been prepared from which the OC curve for selected tests can be obtained (Tang 1938, Pearson and Hartley 1951, Lehmer 1944, Fox 1956, Fix 1949). Most of these tables are entered with a different parameter than ξ. Appendix A contains a

computer program which will calculate the OC curve (as a function of λ) for any given test.

Originally ANOVA was derived from a distributional point of view, but the F-test has been found to possess several optimum properties. Hsu (1941) showed that the F-test is UMP amongst all tests of size α whose power depends upon λ, and Wald (1942a) proved that the F-test is best when one is interested uniformly in all alternatives, as expressed by uniform weighting on spheres. As far as ANOVA is concerned it is immaterial whether the value of λ is built up by a number of small contributions or a single large one. Situations where instead the main emphasis is on detection of large deviations should not use ANOVA since the test is no longer optimum in these cases.

1.2 SEQUENTIAL ONE-WAY FIXED EFFECTS ANALYSIS OF VARIANCE

Wald (1947) first presented, and systematically studied, the sequential test of a simple hypothesis against a simple alternative. Let H_0 denote the hypothesis that the population density is $f_0(x)$, and H_1 the hypothesis that it is $f_1(x)$. Constants A and B are chosen (A > B), and after each observation in a sequence the corresponding likelihood ratio is computed:

$$\Lambda_n = \frac{f_1(x_1) \cdot f_1(x_2) \cdots f_1(x_n)}{f_0(x_1) \cdot f_0(x_2) \cdots f_0(x_n)}$$

The procedure is then as follows: reject H_0 if $\Lambda_n' \geq A$, accept H_0 if $\Lambda_n \leq B$, and obtain another observation if $B < \Lambda_n < A$. A and B are chosen so as to make the probabilities of Type-I and Type-II errors equal to α and β respectively.

Exact values of A and B are difficult to obtain. However, Wald (1947) proved that for small values of α and β

$$A \simeq \frac{1 - \beta}{\alpha} \quad \text{and} \quad B \simeq \frac{\beta}{1 - \alpha}$$

Since the hypothesis about the equality of K normal population means with common unknown variance is a composite multiparameter hypothesis with a nuisance parameter, Wald's theory of the sequential probability ratio test cannot be directly applied. To deal with problems such as these, Wald introduced the method of weight functions which, through the notion of a prior distribution for unknown parameters, essentially reduced the basic problem to test hypotheses in one parameter families. A difficulty with this procedure is the choice of the weight function. Cox (1952) devised a unified method under which sequential tests can be obtained for composite hypotheses. The basic idea behind Cox's procedure is to consider a sequence formed by transforming the original observations, the transformation chosen so that the new sequence depends upon a single parameter. Although the distribution of the transformed values $\{T_n\}$ depends upon only a single para-

meter θ, the sequence $\{T_n\}$ may not be independent. Cox gave conditions under which the following factorization is possible

$$f(T_1,T_2,\cdots,T_n) = f(T_n|\theta)f(T_2,\cdots,T_n)$$

where $f(T_2,\cdots,T_n)$ does not depend upon θ. When this factorization is possible a sequential test can be developed to make a decision about this single parameter θ, using only the transformed values $\{T_n\}$. The test for discriminating between the hypotheses

$$H_0: \ \theta = \theta_o \qquad \text{vs.} \qquad H_1: \theta = \theta_1$$

can now be constructed by considering the following ratio

$$\Lambda_n = \frac{f(T_n|\theta_1)}{f(T_n|\theta_0)} .$$

Johnson (1953) applied Cox's method to the following one-way fixed effects analysis of variance problem. An experiment is carried out in stages, and at each stage a fixed number r_i, for $i = 1,\cdots,k$, of observations are taken from each group. Denote the jth observation on the ith group at the nth stage by X_{ijn}.

Let

$$SSB_n = n \sum_{i=1}^{k} r_i (\bar{X}_i - \bar{\bar{X}})^2$$

and

$$SSW_n = \sum_{i=1}^{k} \sum_{j=1}^{r_i} \sum_{s=1}^{n} (X_{ijs} - \bar{X}_i)^2$$

with

$$\bar{X}_i = \frac{1}{nr_i} \sum_{j=1}^{n_i} \sum_{s=1}^{r} X_{ijs}$$

$$\bar{\bar{X}} = \frac{1}{N} \sum_{i=1}^{k} \sum_{j=1}^{n_i} \sum_{s=1}^{r} X_{ijs}$$

$$N = n \sum_{i=1}^{k} r_i$$

and

$$F_n = \frac{SSB_n/(k-1)}{SSW_n/(N-k)} \tag{1.2.1}$$

The distribution of the sequence $\{F_n\}$ depends only upon the noncentrality parameter λ. Applying Cox's theorem, a sequential test for discriminating between the hypotheses

$$H_0: \ \lambda = \lambda_0 \quad \text{vs.} \quad H_1: \ \lambda = \lambda_1 \ , \ \lambda_1 > \lambda_0 \tag{1.2.2}$$

for a given α and β is specified by the decision rule

$$\text{Accept } H_0 \text{ if } \frac{f(F_n|\lambda_1)}{f(F_n|\lambda_0)} \leq \frac{\beta}{1-\alpha}$$

$$\text{Reject } H_0 \text{ if } \frac{f(F_n|\lambda_1)}{f(F_n|\lambda_0)} \geq \frac{1-\beta}{\alpha}$$

otherwise continue to the next stage. (1.2.3)

An equivalent test was derived by Hoel (1955) using Wald's method of weight functions. The weight function Hoel employed was a generalization of that used for Wald's sequential t-test.

The same sequential test (i.e., the test statistic of (1.2.1) and decision rule (1.2.3) of the hypotheses (1.2.2) has also been by Hall, Wijsman and Ghosh (1965). Their derivation involved applying the principal of invariance. They showed that test statistic of equation (1.2.1) is unchanged by any of the following transformations:

(i) $X'_{ijn} = CX_{ijn} \quad C > 0$

(ii) $X'_{ijn} = X_{ijn} + C$

(iii) an orthogonal transformation

Also, they were able to prove that the sequential test was UMP for testing the hypotheses $H_0: \lambda \leq \lambda_0$ vs. $H_1: \lambda \geq \lambda_1$, by showing that the density $f(F_n|\lambda)$ possessed a monotone likelihood ratio (Lehman (1959)).

In addition, they proved that the vector of statistics $T_n = \{\bar{X}_1, \bar{X}_2, \ldots, SSW_n\}$ was a transitive sufficient sequence. This finding is of importance in later chapters of the thesis.

As previously explained, the sequential test is carried out in stages, where at each stage a fixed number r_i, for $i = 1, \ldots, k$, of observations are taken from each group. Throughout the remainder of this thesis it will be assumed that at the first stage two observations from each group will be taken (this is so the statistic SSB_1 will not be zero on

the first stage). Each subsequent stage will result in one observation from each group being taken (i.e., $r_i = 1$ for all i). All future discussions will pertain to this particular testing situation.

As in the fixed sample test, the density of the statistic F_{nj} $(F_n|\lambda)$, is that of a noncentral F variate and is given in equation (1.1.1). Therefore, the decision rule of equation (1.2.3) requires calculating the ratio of two noncentral F densities. For specified values of α, β, λ_0 and λ_1 the decision rule can be reexpressed as:

$$\text{accept } H_0 \text{ if } \Lambda_n \leq \frac{\beta}{1-\alpha}$$

$$\text{reject } H_0 \text{ if } \Lambda_n \geq \frac{1-\beta}{\alpha}$$

continue otherwise

where

$$\Lambda = R(F_n) = \frac{e^{-\frac{n}{2}(\lambda_1-\lambda_0)}\, M\left[\frac{N-1}{2}, \frac{K-1}{2}, \frac{\lambda_0(K-1)F_n}{2\left(K(n-1) + (K-1)F_n\right)}\right]}{M\left[\frac{N-1}{2}, \frac{K-1}{2}, \frac{\lambda_1(K-1)F_n}{2\left(K(n-1) + (K-1)F_n\right)}\right]}$$

and $M(x,y,u)$, known as the confluent hypergeometric function is defined as

$$M(x,y,u) = \sum_{t=0}^{\infty} \frac{\Gamma(y)\ \Gamma(x+t)}{\Gamma(x)\ \Gamma(y+t)} \frac{u^t}{t!}$$

Since the above decision rule is a function of the statistic, F_n, the equations may be solved to obtain a decision rule in terms of that statistic. That is, two

critical values of the statistic may be found; F_n^A and F_n^R such that $R(F_n^A) = \beta/(1-\alpha)$ and $R(F_n^R) = (1-\beta)/$. When these critical values have been calculated for all stages, F_n^A and F_n^R, $n = 2,\ldots,m_0$; the sequential test can then be conducted by comparing the statistic, F_n, of equation (1.2.1) against these critical values. In summary, at every stage n the following decision rule is applied:

$$\text{accept } H_0 \text{ if } F_n \leq F_n^A$$
$$\text{reject } H_0 \text{ if } F_n \geq F_n^R$$
$$\text{continue if } F_n^A < F_n > F_n^R$$

The test is usually performed using the somewhat simpler statistic

$$V_n = \frac{SSB_n}{SSW_n} \quad .$$

The relationship between the two statistics F_n and V_n is simply

$$\frac{(N-K)V_n}{(K-1)} = F_n \quad .$$

Conducting the test with the statistic V_n requires transforming the critical region as well (e.g., $V_n^A = (K-1)F_n^A/(N-K)$).

Tables of the critical values have been prepared for selected values of α, β, K, λ_0 and λ_1 by Ray (1956) and B.K. Ghosh, et al. (1967). However, these tables are in terms of the test statistic $G_n = V_n/K$. Appendix B of this thesis contains a computer program which calculates the critical values of V_n; V_n^A and V_n^R, for specified values of α, β, K, λ_0, and λ.

As with all statistical tests, one important property of the test described above is the Operating Characteristic Curve. The OC curve for the above test is strictly a function of λ, and is given by

$$OC(\lambda^*) = Pr\{\text{accepting } H_0: \lambda = \lambda_0 \text{ if } \lambda = \lambda^*\}$$

Wald developed an approximation for the OC curve of a sequential probability ratio test of $f(X,\theta_0)$ against $f(X,\theta_1)$ provided the equation

$$E_\theta\{[f(X,\theta_1)/f(X,\theta_0)]^h\} = 1$$

has a nonzero solution $h = h(\theta)$, and the $\{X_1\}$ are i.i.d. However, since the above test is conducted on the transformed sequence $\{V_i\}$ which are not independent, Wald's approximation is not valid. Bhate (1955) developed a conjectural formula, similar to Wald's approximation for the OC curve, when the $\{X_i\}$ are not independent. Ghosh (1970) suggests that substituting the sequence $\{V_i\}$ into Bhate's formula may yield a useful approximation to the OC curve. The result of this substitution yields the following approximation to the OC curve.

If $h_i(\lambda)$ is a nonzero solution of the equation

$$\frac{f_i(V_i|V_1,\cdots,V_{i-1};\lambda_1)}{f_i(V_i|V_1,\cdots,V_{i-1};\lambda_0)}^{h_i} dF(V_i|V_1,\cdots,V_{i-1};\lambda) = 1$$

and $h_i(\lambda) \approx h(\lambda)$ for all $i \geq 1$, that is $h_i(\lambda)$ varies very little with i for a given λ, then

$$OC(\lambda) \underset{\sim}{} \frac{e^{Ah(\lambda)}-1}{e^{Ah(\lambda)}-e^{Bh(\lambda)}} .$$

Where

$$A \underset{\sim}{} \ell n \frac{1-\beta}{\alpha} \qquad B \underset{\sim}{} \ell n \frac{\beta}{1-\alpha}$$

The crucial point in the use of the conjecture lies in the verification of $h_i(0) \underset{\sim}{} h(0)$ for various values of i. Also it must be noted that this approximation is only valid for infinite Wald regions.

The only other alternative, to date, for obtaining the OC curve for this type of test is to employ Monte Carlo techniques.

Also of interest in a sequential test is the Average Sample Number function. For the above test the ASN function will be defined as:

$ASN(\lambda^*)$ = Expected number of stages until a decision is reached if $\lambda = \lambda^*$.

As with the OC curve, Wald's approximation to the ASN, is not valid due to the dependence of the $\{V_i\}$ sequence. No general formula (exact or approximate) for the ASN for composite hypotheses exists, but Bhate (1955) has developed

a conjectural formula along the same lines as that for the OC curve. Ray (1956) has applied Bhate's conjectural formula to the one-way fixed effect analysis of variance test, and obtained expressions for $\lambda = \lambda_0, \lambda_1$. Again, as with the OC curve this procedure is valid only for open Wald regions.

Since the regions are open, it is possible to progress through a large number of stages before a decision is reached. The number of stages will always be finite, however (Johnson, 1953). One way of assuming termination within a reasonable amount of time is to truncate the test. Truncation involves altering the Wald regions so that by some stage m_0 a decision can be made.

This thesis will be concerned with developing procedures to obtain the ASN function and OC curve for a SANOVA test with any given set of truncated regions. The following chapter contains a derivation of SANOVA for the case $k = 2$ by the Direct Method of Sequential Analysis (Aroian, 1968).

1.3 CONCLUSION

This chapter has served to introduce the SANOVA test. This thesis will pertain to obtaining the OC and ASN functions of such a test. Currently, only approximations exist, such as that of Bhate (1959), considered in this chapter. The next chapter will derive the first exact procedure for obtaining the OC and ASN of a k=2 SANOVA test.

2.0 INTRODUCTION

The major advantage to performing an analysis of variance sequentially is the possible reduction in sample size over that required for the fixed sample test. Since the sample size is not predetermined in a SANOVA test, the experimenter would like to be assured that the sequential test can offer an equally discriminating test with smaller sample size than the corresponding fixed sample test. As previously discussed, such assurance can be obtained by examining the OC and ASN curves of the sequential test.

In this chapter an exact procedure is developed for obtaining the OC and ASN curves of a SANOVA test. This procedure is the first which yields exact results and is versatile enough so as to be used for tests with general regions. It is hoped that the procedure will be an invaluable tool for designing SANOVA tests.

In a SANOVA test the decision of acceptance can be made at any stage i, $i = 2, \ldots, m_0$. Thus, the probability of accepting the null hypothesis must be calculated as the sum of the probabilities of accepting at each state, $P_A^{\,i}$; i.e. $OC = \sum_{i=2}^{m_0} P_A^{\,i}$. Of course, these probabilities will depend upon the state of nature λ.

Unlike the fixed sample test, these probabilities cannot be obtained by simply integrating the distribution of the test statistic. One must remember that in sequential analysis the statistic at stage i only exists when the statistics at all previous stages have had values within the continuation region, i.e., $F_A^j < F_j < F_R^j$, $j = 2, \ldots, i-1$. Thus, the distribution of the test statistic at stage i is not a true probability distribution since its total probability content is not 1 (rather P_c^{i-1}). Were this distribution known it could be integrated to obtain P_A^i. Unfortunately this distribution cannot be obtained analytically.

However, the procedure developed in this chapter obtains a "truncated" density at stage i-1. Rather than utilizing the density of the test statistic F_i, this procedure utilizes the joint density of the sufficient statistics at stage i (i.e. each of the K sample means and the pooled estimate of the variance). From this joint density the density of F_i can be obtained which then can be integrated to yield P_A^i.

The joint density at stage i is obtained from the joint density at stage i-1 by applying Aroian's direct method of sequential analysis. This consists of determining the mapping of points at stage i-1 to those at

stage i (where a point represents a value of the vector of sufficient statistics). This mapping describes how the statistics at stage i-1 are changed by the new observations to yield statistics at stage i. Thus for any given point, A, at stage i, there is a region of points, P, at stage i-1 which can be mapped into it.

Due to the nature of a sequential test, some points in P may result in a decision being made at stage i-1. If so, the point can not be mapped into A, since the test would terminate at stage i-1. Thus, for a sequential test the region of points, H, which can be mapped into A must include only those points in P which lie in the continuation region at stage i-1. Ultimately, this restriction will yield the desired "truncated" density for stage i (i.e., the total probability content is P_c^{i-1}).

As previously mentioned the statistics at stage i are transformations of the statistics at stage i-1 and the new observations taken at stage i. Suppose that the required number of sufficient statistics is n, and that the number of new observations taken at any stage is K (assuming one observation from each population would imply a test for the equality of K means). The above transformation would then be a transformation of n + k random variables (the statistics at stage i and the K new

observations) to n random variables (the statistics at stage i). Since the dimensionality of the two sets of random variablesis not the same, K surplus random variables must be introduced. These K surplus variables will be judiciously selected functions of the statistics at stage $i-1$ and new observations. This introduction of surplus random variables makes the transformation from an $n + K$ dimensional space (the statistics at stage $i-1$ and K new observations) to an $n+K$ dimensional space (the statistics at stage i and K surplus variables). The joint density of the statistics at stage i and K surplus variablescan be found by calculus. The procedure is essentially equivalent to transforming variables in multiple integrals.

Finally, the desired density (of the joint distribution of the statistics at stage i) is obtained by performing a multiple integration of the joint density of the statistics at stage i and K surplus variables. The region of integration will be over all points contained in the set of points H.

The above discussion has given a brief outline of the "exact" procedure developed in this chapter of the thesis. The following sections describe the procedure in greater detail.

2.1 THE DIRECT METHOD OF SEQUENTIAL ANALYSIS

Aroian developed a general theory for obtaining the properties of a sequential test exactly (Aroian, 1968).

To determine the properties (usually only the OC curve) for a fixed sample test one needs to know the distribution of the test statistic for a given sample size n for different values of the parameter being tested. For example, in the fixed sample analysis of variance test where n observations are taken from each of k groups, the test statistic

$$F_{cal} = \frac{\sum_{i=1}^{k} (\bar{X}_i - \bar{\bar{X}})^2/(k-1)}{\sum_{i=1}^{k} \sum_{j-1}^{n} (X_{ij} - X_i)^2/N-k)}$$

is distributed as a noncontrol F variate with [k-1,N-k] degrees of freedom and noncentrality parameter $\xi = n\lambda$. The OC curve for a given value of the parameter, ξ, is then obtained by integrating this distribution over the acceptance region. For fixed sample ANOVA

$$OC(\xi) = \int_{\text{Acceptance region}} f_{k-1,N-k,\xi}(F_{cal})\, dF_{cal}$$

where $f_{v_1,v_2,\xi}(X)$ is the noncentral F density function.

The direct method recognizes as its primary principle that observations are taken in stages in sequential testing, and that for this reason a way must be found to calculate the distribution of the test statistic T_n, at stage n. In most cases T_n is not independent of $T_1, T_2, \cdots, T_{n-1}$, so that the marginal distribution of T_n must be obtained by integrating the joint distribution, i.e.,

$$h_n(T_n) = \int \cdots \int_I h(T_1, T_2, \cdots, T_{n-1}) dT_1 + T_2 \cdots dT_{n-1}$$

Since a sequential test is terminated whenever any $T_m \leq T_m^A$ or $T_m \geq T_m^R$; it is not possible to have a value of T_n if any $T_i \leq T_i^A$ or $T_i \geq T_i^R$ $i=2,\cdots,n-1$. Therefore, the direct method considers only the truncated distribution $f_n(T_n)$, where

$$f_n(t) = \Pr\{T_2^A < T_2 < T_2^R, T_3^A < T_3 < T_3^R, \cdots T_{n-1}^A < T_{n-1} < T_{n-1}^R, T_n = t\}.$$

Mathematically f_n is not a true "density" function since $\int f_n \neq 1$, but will still be referred to as a density.

When T_n is dependent upon T_{n-1} in the following manner

$$T_n = g_1(T_{n-1}) + g_2(X_{(n)})$$

with $X_{(n)}$ representing the new observation at stage n, g_1 and g_2

arbitrary functions, $f_n(T_n)$ can be obtained from $f_n(T_{n-1})$. Bahadur generalized the dependence by introducing the notion of a transitive sequence of statistics (Bahadur, 1954). A transitive sufficient sequence $\{T_n\}$ is a sequence such that for every $n > 1$ the conditional distribution of T_{n+1}, given the set of observations up to stage n, is identical to the conditional distribution of T_{n+1}, given T_n. So, in general, whenever T_n is transitive sufficient, $f_n(T_n)$ can be obtained from $f_{n-1}(T_{n-1})$.

Instead of obtaining $f_n(T_n)$ via integration of a joint distribution, the direct method obtains $f_n(T_n)$ directly from $f_{n-1}(T_{n-1})$, due to the transitivity of T_n.

At each stage n, the direct method calculates the probability of accepting H_0, P_A^n, and the probability of rejecting H_0, P_R^n, by integrating $f_n(T_n)$ over the appropriate regions. In mathematical terms,

$$P_A^n = \int_{T_n \leq \underline{T}_n} f_n(T_n)\, dT_n$$

$$P_R^n = \int_{T_n \geq \underline{T}_n} f_n(T_n)\, dT_n.$$

These probabilities depend upon the state of nature θ, since the distribution of T_n depends upon the parameter θ. So for any given θ, the OC and ASN curves may be calculated as:

$$OC(\theta) = \sum_{i=2}^{m_0} P_A{}^i$$

$$ASN(\theta) = \sum_{i=2}^{m_0} i(P_A{}^i + P_R{}^i) = 1 + \sum_{i=1}^{m} P_C{}^i$$

where m_0 is the truncation point of the sequential test.

Usually the density $f_n(T_n)$ cannot be obtained from $f_{n-1}(T_{n-1})$ analytically, so that the procedure must be performed numerically. In numerical terms $f_n(T_n)$ represents a "grid" of T_n values calculated for each n from a "grid" of T_{n-1} values.

The direct method has been used in a variety of applications, including tests for the mean of a normal distribution with the standard deviation known (Aroian and Robison, 1969) and unknown (Schmee, 1974), and tests of the standard deviation of a normal distribution with mean known and unknown (Aroian, Gorge, Goss and Robison, 1975).

The following section contains a discussion of the application of the direct method to SANOVA.

2.2 APPLICATION OF THE DIRECT METHOD TO SANOVA

SANOVA is based on the statistic

$$V_n = n \sum_{i=1}^{k} (\bar{X}_{i(n)} - \bar{\bar{X}}_{(n)})^2 / \sum_{i=1}^{k} \sum_{j=1}^{n} (X_{ij} - \bar{X}_{i(n)})^2 .$$

In order to solve this problem by the direct method a transitive, sufficient sequence $\{T_n\}$ must be used. The sequence $\{V_n\}$ is not transitive, so one must use the multidimensional transitive sequence

$$\{T_n\} = \{X_{1(n)}, X_{2(n)}, \cdots, X_{k(n)}, S^2_{(n)}\}$$

where

$$X_{i(n)} = \sum_{j=1}^{n} X_{ij}$$

and

$$S_{(n)}^2 = \sum_{i=1}^{k} \sum_{j=1}^{n} \left[X_{ij} - \frac{X_{i(n)}}{n} \right]^2$$

(Hall, Wijsman, Ghosh, 1965). Similarly, one must now work with the joint distribution $f_n(X_{1(n)}, X_{2(n)}, \cdots X_{k(n)}, S_{(n)}^2)$. From this distribution P_A^n and P_R^n can be obtained by a k+1 dimensional integration,

$$P^n = \iiint_A \cdots \int f_n(X_{1(n)}, X_{2(n)}, \cdots, S^2_{(n)}) dX_{1(n)} \cdots dS^2_{(n)}$$

where A is the region in k+1 space such that

$$V_n = \sum_{i=1}^{k} \left\{ X_{i(n)} - \left[\frac{X_{1(n)} + \cdots + X_{k(n)}}{k} \right] \right\} / S^2(n) \leq V_n^L$$

and

$$P_R^n = \int \cdots \int_R \cdots \int f_n(X_{1(n)}, \cdots, X_{k(n)}, S^2(n)) dX_{1(n)} \cdots dS^2(n)$$

where R is the region such that $V_n \geq V_n^u$.

The problem lies in obtaining $f_n(T_n)$. If the first stage at which a decision can be made is $n_1 \geq 2$, then since $X_{1(n_1)}, X_{2(n_1)}, \cdots, X_{k(n_1)}$, and $S^2_{(n_1)}$ are all independent

$$f_{n_1}(T_{n_1}) = f_{n_1}(X_{1_{(n1)}}, X_{2_{(n_1)}}, \ldots, X_{K_{(n_1)}}, S_{(n_1)}^2)$$

$$= \phi\left[\frac{X_{1(n_1)} - n_1\mu_1}{\sigma}\right] \cdot \phi\left[\frac{X_{2(n_1)} - n_1\mu_2}{\sigma}\right] \cdots \phi\left[\frac{X_{k(n_1)} - n_1\mu_2}{\sigma}\right]$$

$$\sigma^2 f_{\chi^2 k(n_1-1)}(S^2_{(n_1)})$$

where

$$\phi(X) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}X^2}$$

is the standard normal density function and

$$f_{\chi^2_v}(X) = \frac{X^{v/2-1}e^{-X/2}}{2^{v/2}\Gamma(v/2)}, \quad X \geq 0$$

is the χ^2 density function with v degrees of freedom. Since the power of the SANOVA test depends only upon λ, the density $f_{n_1}(T_{n_1})$ for given λ can be calculated by assuming $\mu_1 = \mu_2 = \cdots = \mu_k$, $\sigma = 1$ and

$$\mu_k = \sqrt{\frac{\lambda^*}{k-1}} .$$

The probabilities $P_A^{n_1}$ and $P_R^{n_1}$ need not be obtained by integration of $f_{n_1}(T_{n_1})$ since the distribution of V_{n_1} is known to be related to the noncentral F-distribution;

$\frac{k(n_1-1)}{(k-1)}V_{n_1} \sim F_{k-1,k(n_1-1)}(n_1\lambda^*)$. Therefore

$$P_A^{n_1} = \int_0^{\frac{k(n_1-1)}{(k-1)}V_{n_1}^{A}} f_{k-1,k(n_1-1),n,\lambda^*}(X)\, dX$$

and

$$P_R^{n_1} = \int_{\frac{k(n_1-1)V_{n_1}^{R}}{(k-1)}}^{\infty} f_{k-1,k(n_1-1),n_1\lambda^*}(X)\, dX$$

These integrals are evaluated by the methods discussed in Appendix A.

To determine $f_{n_1+1}(T_{n_1+1})$ the direct method will be applied. Since $\{T_n\}$ is transitive, $f_{n_1+1}(T_{n_1+1})$ can be obtained directly from $f_{n_1}(T_{n_1})$. Suppose the following relationships exist between the elements of T_{n_1+1} and T_{n_1};

$$\begin{aligned} X_{1(n_1+1)} &= g_{11}(X_{1(n_1)}) + g_{21}(X_{(n_1+1)}) \\ &\vdots \\ X_{k(n_1+1)} &= g_{1k}(X_{k(n_1)}) + g_{2k}(X_{(n_1+1)}) \end{aligned}$$

$$S^2_{(n_1+1)} = g_{1k+1}(S^2_{(n_1)}) + g_{2k+1}(X_{(n_1+1)})$$

where g_{1i} and $g_{2i}, i=1,\cdots,k+1$ are arbitrary functions, and $X_{(n_1+1)}$ is the vector of new observations from stage n_1+1. The statistic T_{n_1+1} defines a transformation which maps points in the 2k+1 dimensional space of $T_{n_1}, X_{(n_1+1)}$ to the K+1 dimensional space of T_{n_1+1}. To make the transformation from a 2k+1 space to a 2k+1 space, the following additional variables will be defined

$$\begin{aligned} E_{1(n_1+1)} &= g_{2\ k+2}(X_{(n_1+1)}) \\ &\vdots \qquad\qquad\qquad\qquad : E_{n_1+1} \\ E_{k(n_1+1)} &= g_{2\ 2k+1}(X_{(n_1+1)}) \end{aligned}$$

where the functions g_{2i}, $i=k+2,\cdots,2k+1$ are arbitrary functions. Since the transformation is now 2k+1 to 2k+1, the joint distribution of $X_{1(n_1+1)},\cdots,E_{k(n_1+1)}$ can be obtained. From this distribution the joint marginal distribution of $X_{1(n_1+1)},\cdots,S^2_{(n_1+1)}$ will be obtained by integrating out $E_{1(n_1+1)},\cdots,E_{k(n_1+1)}$. To obtain the joint distribution of T_{n_1+1} and E_{n_1+1}, one must first obtain the joint distribution of T_{n_1} and $X_{(n_1+1)}$. Since $X_{(n_1+1)}$ is independent of T_{n_1}, the joint distribution is simply the product of the respective densities; i.e.

$$g\left(X_{1(n_1)},\cdots,S^2_{(n_1)},X_{(n_1+1)}\right) = f_{n_1}\left(X_{1(n_1)},\cdots,S^2_{(n_1)}\right) \cdot f\left(X_{(n_1+1)}\right)$$

Then under certain conditions (which for this general discussion will be assumed to be true, but are dependent upon the functions g_{1i} and g_{2i}) the joint distribution of T_{n_1+1} and E_{n_1+1} is given by

$$f\left(T_{n_1+1},E_{n_1+1}\right) = g\left(u_1\left(T_{n_1},X_{(n_1+1)}\right),\cdots,u_{2k+1}\left(T_{n_1},X_{(n_1+1)}\right)\right)|J|$$

where $u_i\left(T_{n_1}, X_{(n_1+1)}\right), i=1,\cdots,2k+1$ is the set of inverse transformations and $|J|$ is the Jacobian for the transformation. As previously mentioned the density of $f_{n_1}(T_{n_1+1})$ can now be obtained as follows

$$f_{n_1}\left(T_{n_{1+1}}\right) = \iint\cdots\int_{R^*} f\left(T_{n_1+1}, E_{n_1+1}\right) dE_{1(n_1+1)} \cdots dE_{k(n_1+1)} \cdot$$

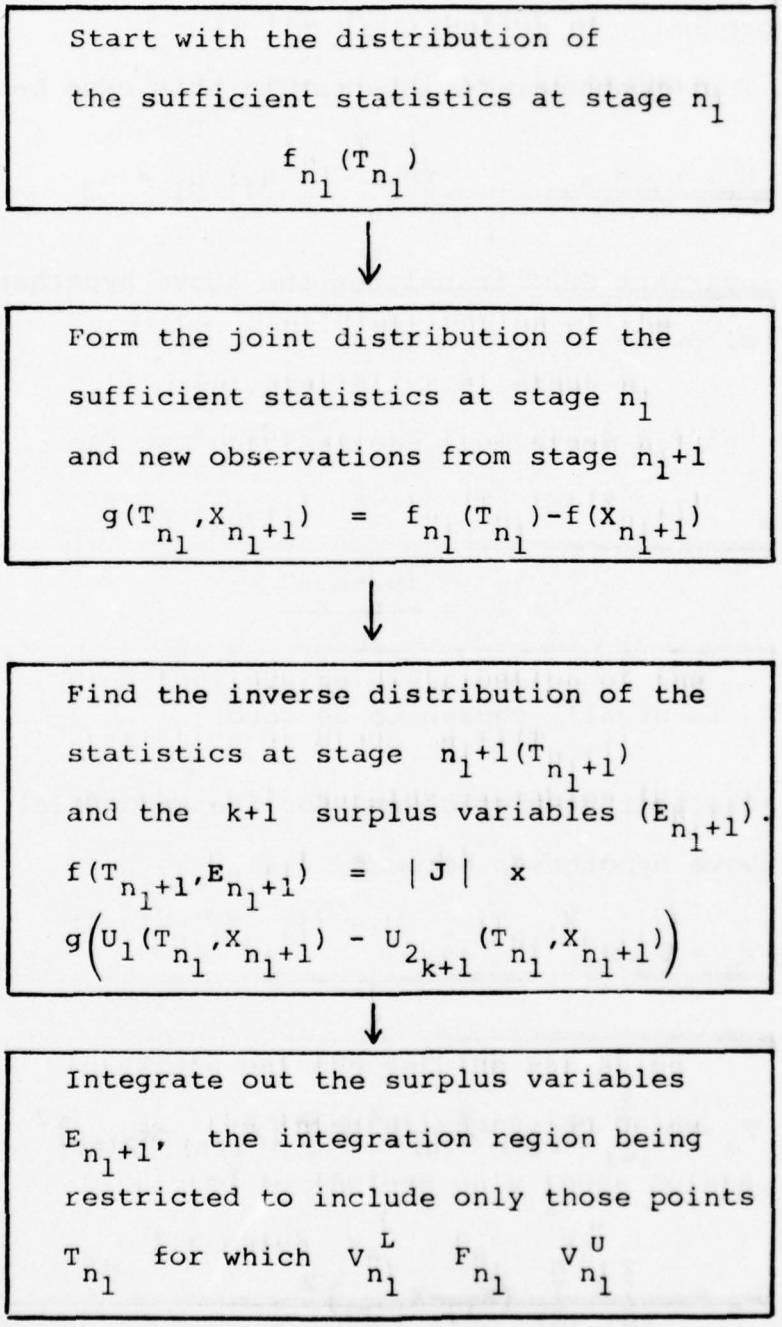
where R^* represents the integration region in k space.

The direct method restricts the set of points $T_{n_1}, X_{(n_1+1)}$ to be mapped into T_{n_1+1}, to include only those points for which $V_{n_1}{}^L < V_{n_1} < V_{n_1}{}^u$. This entire procedure can be represented diagramatically as shown in Figure 1.

The following section contains a complete derivation of the direct method procedure to obtain $f_n(T_n)$ from $f_{n-1}(T_{n-1})$ for the special case k=2. This discussion will specify the functions g_{1i}, g_{2i}, u_i and derive the integration region R^*.

FIGURE 1

The Direct Method Logic



2.3 DERIVATION FOR THE CASE k = 2

This section will derive a procedure for obtaining the properties of a SANOVA test for the special case when k=2, groups.

The hypotheses of interest in this case become:

$$H_0: \mu_1 = \mu_2 \qquad \text{vs.} \qquad H_1: \mu_1 \neq \mu_2$$

The invariant SPRT translates the above hypotheses into the following

$$H_0: \lambda \leq \lambda_0 \qquad H_1: \lambda \geq \lambda_1$$

where

$$\lambda = \frac{(\mu_1-\mu_2)^2}{2\sigma^2}$$

and λ_0 is usually chosen to be zero.

The test statistic used for the sequential test of the above hypotheses becomes

$$V_n = T_n/D_n$$

where

$$T_n = n \sum_{i=1}^{2} (\bar{X}_{i(n)} - \bar{\bar{X}}_{(n)})^2 = \frac{n}{2} [X_{1(n)} - X_{2(n)}]^2$$

and

$$D_n = \sum_{i=1}^{2} \sum_{j=1}^{n} (X_{ij} - X_{i(n)})^2 .$$

To conduct such a test requires the specification of the following quantities: the null hypothesis, λ_0; the alternative hypothesis, λ_1; and a set of regions V_A^i, V_R^i, $i=1,\cdots,m_0$ (m_0 being the test truncation point). The regions may be any type (Wald or modified Wald) that specify: accept H_0 if at any i, $V_i \leq V_A^i$ and reject H_0 if $V_i \geq V_R^i$; otherwise continue sampling. The properties of such a test consist of the OC and ASN curves as functions of λ; i.e., OC (λ), and ASN (λ), $\lambda_0 \leq \lambda \leq \lambda_1$.

The direct method involves calculating for a given λ^*, $f_n(T_n)$ at each stage n, from which the probabilities P_A^n and P_R^A are obtained. Once the quantities P_A^i, P_R^i, $i=1,\cdots m_0$ have been calculated, the points $OC(\lambda^*)$ and $ASN(\lambda^*)$ may be obtained. The following discussion will pertain to obtaining P_A^i, P_R^i and thus the OC and ASN for a given state of nature $\lambda = \lambda^*$. Unfortunately, the statistic V_n is not transitive, and in order to conserve all the necessary information, one must resort to a transitive sufficient sequence, such as $\{T_n\} = \{W_n, Q_n, R_n\}$ where

$$W_n = \sum_{j=1}^{n} X_{1_j}$$

$$Q_n = \sum_{j=1}^{n} X_{2_j}$$

$$R_n = \sum_{i=1}^{n} \sum_{j=1}^{n} X_{1_j}^2 .$$

The reduction from $T_n \rightarrow V_n$ is performed at each stage in the following manner

$$V_n = \frac{\left[W_n - Q_n\right]^2}{2\left[nR_n - W_n^2 - Q_n^2\right]} \cdot \quad (2.3.1)$$

The direct method involves calculating, for every stage n, the joint density $f_n(W_n, Q_n, R_n)$.

Suppose the first stage at which a decision can be made is $n_1 \geq 2$. The density of $f_{n_1}(W_{n_1}, Q_{n_1}, R_{n_1})$ is obtained as follows:

Let

$$W_{n_1} = n_1 X$$

$$Q_{n_1} = n_1 y$$

$$R_{n_1} = D_{n_1} + n_1 X^2 + n_1 y^2$$

where

$$X = \bar{X}_{1(n_1)}$$

$$Y = \bar{X}_{2(n_1)}$$

$$D = \sum_{j=1}^{2} \sum_{i=1}^{n_1} \left(X_{ij} - \bar{X}_{i(n_1)}\right)^2 .$$

Since the quantities X, Y, D are all independent, their joint distribution is given by:

$$f(X,Y,D) = \left[\chi^2_{2(n_1-1)}(D_{n_1})\cdot\sigma^2\right]\phi\left[\frac{\bar{X}_{1(n_1)}-\mu_1}{\sigma/\sqrt{n_1}}\right]\cdot\phi\left[\frac{\bar{X}_{2(n_1)}-\mu_2}{/\sqrt{n_1}}\right].$$

Since this procedure is being used to find the properties of the test when $\lambda = \lambda^*$, and the test is invariant with respect to λ, we can let $\mu_1 = 0$, $\sigma = 1$, and

$$\mu_2 = \sqrt{\lambda^*}\ .$$

So this density may be expressed as

$$f(X,y,D_{n_1}) = \chi^2_{2(n_1-1)}(D_{n_1})\cdot\phi\left(\sqrt{n_1}\,\bar{X}_{1(n_1)}\right)\cdot\phi\left(\sqrt{n_1}\,(X_{2(n_1)}-\sqrt{\lambda^*})\right).$$

From this density we can determine the joint distribution of $W_{n_1}, Q_{n_1}, R_{n_1}$. The set of inverse transformations is given by

$$X = \frac{1}{n_1} W_{n_1}$$

$$Y = \frac{1}{n_1} Q_{n_1}$$

$$D = R_{n_1} - \frac{1}{n_1} W_{n_1}^2 - \frac{1}{n_1} Q_{n_1}^2$$

which has a Jacobian

$$J = \begin{vmatrix} \frac{1}{n_1} & 0 & 0 \\ 0 & \frac{1}{n_1} & 0 \\ -\frac{2}{n_1} W_{n_1} & -\frac{2}{n_1} Q_{n_1} & 1 \end{vmatrix} = \frac{1}{n_1^2}$$

Since this transformation is one-to-one

$$f_{n_1}(W_{n_1},Q_{n_1},R_{n_1}) = \left(\frac{1}{n_1 2}\right)\chi^2_{2(n_1-1)}\left[R_{n_1} - \frac{1}{n_1}W_{n_1}^2 - \frac{1}{n_1}Q_{n_1}^2\right]$$
$$\cdot \phi\left(\sqrt{n_1}\left(\frac{1}{n_1}W_{n_1}\right)\right) \cdot \phi\left(\sqrt{n_1}\left(\frac{1}{n_1}Q_{n_1} - \sqrt{\lambda^*}\right)\right).$$

From this density, $F_{n_2}(W_{n_2},Q_{n_2},R_{n_2})$ will be obtained, where $n_2 = n_1+1$.

First consider the following functional relationships between the statistics at stage n_1 and stage n_2:

$$W_{n_2} = W_{n_1} + X_{1n_2} \qquad (2.3.2)$$
$$Q_{n_2} = Q_{n_1} + X_{2n_2}$$
$$R_{n_2} = R_{n_1} + X^2_{1n_2} + X^2_{2n_2}$$

The statistics are changed from stage n_1 to stage n_2 by two new observations, X_{1n_2} from group 1 and X_{2n_2} from group 2. Since X_{1n_2} and X_{2n_2} are independent of W_{n_1}, Q_{n_1}, R_{n_1}, the joint distribution of X_{1n_2}, X_{2n_2}, W_{n_1}, Q_{n_1}, R_{n_2} is simply

$$f^P_{n_1}(W_{n_1},Q_{n_1},R_{n_1},X_{1n_2},X_{2n_2}) = f_{n_1}(W_{n_1},Q_{n_1},R_{n_1}) \cdot \phi(X_{1n_2}) \cdot \phi(X_{1n_2} - \sqrt{\lambda^*}).$$

Equations (2.3.2) represent a transformation from the 5 dimensional space of W_{n_1}, Q_{n_1}, R_{n_1}, X_{1n_2}, X_{2n_2} to the 3 dimensional space of W_{n_2}, Q_{n_2}, R_{n_2}. A transformation from

5 dimensional space to 5 dimensional space can be achieved by introducing the surplus variables Z and U, yielding the following transformation, T:

$$
\begin{aligned}
W_{n_2} &= W_{n_1} + X_{1n_2} \qquad (2.3.3)\\
Q_{n_2} &= Q_{n_1} + X_{2n_2}\\
R_{n_2} &= R_{n_1} + X^2_{1n_2} + X^2_{2n_2}\\
Z &= X_{1n_2}\\
U &= X_{2n_2}
\end{aligned}
$$

The set of inverse transformations, T^{-1}, is then given by

$$
\begin{aligned}
W_{n_1} &= W_{n_2} - Z \qquad (2.3.4)\\
Q_{n_1} &= Q_{n_2} - U\\
R_{n_1} &= R_{n_2} - Z^2 - U^2\\
X_{1n_2} &= Z\\
X_{2n_2} &= U
\end{aligned}
$$

This transformation has a Jacobian matrix of the following form

$$
J = \begin{bmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & -2Z & -2U & 1 \end{bmatrix} = \begin{bmatrix} I & 0\\ B_{21} & B_{22} \end{bmatrix}
$$

so that

$$|J| = |I| \; |B_{22}-B_{21}I \; 0| = |I| \; |B_{22}| = |B_{22}| = 1.$$

Thus, the joint distribution of W_{n_2}, Q_{n_2}, R_{n_2}, Z, U is given by

$$f_{n_2}(W_{n_2}, Q_{n_2}, R_{n_2}, Z, U) = f^{P}_{n_1}(W_{n_2}-Z, Q_{n_2}-U, R_{n_2}-Z^2-U^2, Z, U) . \tag{2.3.5}$$

The marginal joint distribution of W_{n_2}, Q_{n_2}, R_{n_2} is obtained by integrating (2.3.5) with respect to U and Z over the appropriate regions. Ordinarily this region consists of all possible values of U and Z, $-\infty < U < \infty$, $-\infty < Z < \infty$; so that the marginal is obtained by

$$f(W_{n_2},Q_{n_2},R_{n_2}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{n_1}{}^{P}(W_{n_2}-Z, Q_{n_2}-U, R_{n_2}-Z^2-U^2, Z, U)\,dZdU$$

By substitution this integration becomes

$$f\left(W_{n_2}, Q_{n_2}, R_{n_2}\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{ \frac{1}{n_1^2}\ \chi^2_{2(n_1-1)}\left[R_{n_2}-Z^2 - U^2 - \frac{1}{n_1}\left(W_{n_2}-Z\right)^2 - \frac{1}{n_1}\left(Q_{n_2}-U\right)^2\right]\right.$$

$$\cdot\ \phi\left[\sqrt{n_1}\ \frac{1}{n_1}\left(W_{n_2}-Z\right)\right]\ \cdot\ \phi\left[\sqrt{n}\ \frac{1}{n}\left(Q_{n_2}-U\right)\right] - \sqrt{\lambda^*}$$

$$\left.\cdot\ \phi\left[(Z)\cdot\phi\left(U-\sqrt{\lambda^*}\right)\right]\right\} dZdU$$

Since the chi-squared density function is only defined for positive values, the integration region of U and Z must be chosen so that

$$R_{n_2}-Z^2-U^2-\frac{1}{n_1}\left(W_{n_2}-Z\right)^2-\frac{1}{n_1}\left(Q_{n_2}-U\right)^2\ \geq\ 0$$

Therefore, for a given value of U, say U*, the range of allowable Z values is given by the following roots.

$$Z_{\text{limits}} = \frac{W_{n_2}}{n_1} \pm \sqrt{\frac{n_1}{n_1+1}\left[R_{n_2} - \frac{W_{n_2}^2}{n_1+1} - \frac{Q_{n_2}^2}{n_1+1}\right] - \left(U^* - \frac{Q_{n2}}{n_1+1}\right)^2}$$

(2.3.6).

Let the smaller root (the lower limit of Z integration) be denoted Z_L and the larger (the upper limit of Z) by Z_U.

The limits of the U integration are given by:

$$U_{limits} = \frac{Q_{n_2}}{n_1} \pm \sqrt{\frac{n_1}{n_1+1}\left[R_{n_2} - \frac{W_{n_2}^2}{n_1+1} - \frac{Q_{n_2}^2}{n_1+1}\right]} \qquad (2.3.7).$$

Let the smaller root (the lower limit of U integration) be denoted by U_L and the larger by U_U (the upper limit of U).

It should be noted that equations (2.3.6) and (2.3.7) have solutions only if $R_{n_2} - \frac{W_{n_2}^2}{n_1+1} - \frac{Q_{n_2}^2}{n_1+1} \geq 0$. If this is not the case, all the above limits can be regarded as zero, so that $f(W_{n_2}, Q_{n_2}, R_{n_2}) = 0$.

For all points R_{n_2}, W_{n_2}, Q_{n_2}, such that $R_{n_2} - \frac{W_{n_2}^2}{n_1+1} - \frac{Q_{n_2}^2}{n_1+1} \geq 0$, the joint density is obtained by more

$$f(W_{n_2},Q_{n_2},R_{n_2}) = \int_{U_L}^{U_U}\int_{Z_L}^{Z_U} f_{n_1}{}^{P}(W_{n_2}-Z, Q_{n_2}-U, R_{n_2}-Z^2-U^2, Z, U)\,dZdU. \qquad (2.3.8)$$

The result of this integration yields

$$f(W_{n_2}, Q_{n_2}, R_{n_2}) = \frac{1}{(n_1+1)^2}\, \chi^2_{2(n_1)}\left[R_{n_2} - \frac{1}{(n_1+1)} W_{n_2}^2 - \frac{1}{(n_1+1)} Q_{n_2}^2\right]$$

$$\phi\left[\sqrt{n_1+1}\left(\frac{1}{n_1+1} W_{n_2}\right)\right] \cdot \phi\left(\sqrt{n_1+1}\left(\left(\frac{1}{n_1+1} \left(Q_{n_2} - \sqrt{\lambda^*}\right)\right)\right)\right) .$$

This is the density which results if the first step at which a decision can be made is $n_2 = n_1 + 1$.

However, the direct method restricts the set of points $(W_{n_1}, Q_{n_1}, R_{n_1}, X_{2n_2}, X_{2n_2})$ to consist of only those points such that $V_{n_1}^A < V_{n_1} < V_{n_1}^R$. The limits in equation (2.3.8) do not consider this restriction. The result of applying this restriction involves altering the U and Z limits of integration. The integration region consists of all point U,Z such that:

$$(1) \qquad R_{n_2} - Z^2 - U^2 - \frac{1}{n_1}\left(W_{n_2}-Z\right)^2 - \frac{1}{n_1}\left(Q_{n_2}-U\right)^2 \geq 0$$

$$(2) \qquad V_A{}^{n_1} < \frac{\left[W_{n_2}-Z-Q_{n_2}+U\right]^2}{2\left[n_1\left(R_{n_2}-Z^2-U^2\right)-\left(W_{n_2}-Z\right)^2-\left(Q_{n_2}-U\right)^2\right]} < V_R{}^{n_1}$$

From these constraints integration limits U_U, U_L and Z_U, Z_L can be obtained, such that

$$f_{n_2}\left(W_{n_2}, Q_{n_2}, R_{n_2}\right) = \int_{U_L}^{U_U}\int_{Z_L}^{Z_U} f_{n_1}{}^{P}\left(W_{n_1}-Z,\ Q_{n_2}-U,\ R_{n_2}-U^2-Z^2,\ Z, U\right) dZ\,dU.$$

Explicit expressions for these limits can be best derived geometrically.

Let $V_A = V_A{}^{n_1}$ and $V_R = V_R{}^{n_1}$ such that at stage n_1 H_0 is accepted if

$$\frac{\left[W_{n_1}-Q_{n_1}\right]^2}{2\left[n_1 R_{n_1}-W_{n_1}{}^2-Q_{n_1}{}^2\right]} \leq V_A \qquad (2.3.9)$$

and H_0 is rejected if

$$\frac{\left[W_{n_1}-Q_{n_1}\right]^2}{2\left[n_1 R_{n_1}-W_{n_1}{}^2-Q_{n_1}{}^2\right]} \geq V_R \cdot \qquad (2.3.10)$$

Solving the above expressions, when the equalities are satisfied, yields the following two surfaces:

$$B_A: \quad R_{n_1} = \frac{(2V_A+1)W_{n_1}{}^2+(2V_A+1)Q_{n_1}{}^2-2W_{n_1}Q_{n_1}}{2n_1V_A}$$

$$R_{n_1} = C_A W_{n_1}{}^2+C_A Q_{n_1}{}^2-2P_A W_{n_1}Q_{n_1}$$

and

$$B_R: \quad R_{n_1} = \frac{(2V_R+1)W_{n_1}{}^2+(2V_R+1)Q_{n_1}{}^2-2W_{n_1}Q_{n_1}}{2n_1V_R}$$

$$R_{n_1} = C_R W_{n_1}{}^2+C_R Q_{n_1}{}^2-2P_R W_{n_1}Q_{n_1}$$

where

$$C_A = \frac{2V_A+1}{2n_1V_A} \quad \text{and} \quad P_A = \frac{1}{2n_1V_A}$$

with similar expressions for C_R and P_R.

The surface B_A is an elliptic paraboloid since the discriminant, D

$$D = 4P_A{}^2-4C_A{}^2 = 4\left(1-(2V_A+1)^2\right)$$

will always be negative for $V_A > 0$.

Similarly the surface B_R is an elliptic paraboloid, usually containing the surface B_A. All points lying between these two surfaces constitute the continuation region, C_{n_1}.

Next consider the surface induced by the transformation T. This surface contains all points in T_{n_1} space, $(W_{n_1}, Q_{n_1}, R_{n_1})$, which can be mapped into some point in T_{n_2} space $(W_{n_2} = a,\ Q_{n_2} = b,\ R_{n_2} = c)$.

Since

$$a = W_{n_1} + X_{1n}$$

$$b = Q_{n_1} + X_{2n_1}$$

$$c = R_{n_1} + X_{1n_1}^2 + X_{2n_1}^2 ;$$

this surface is given by

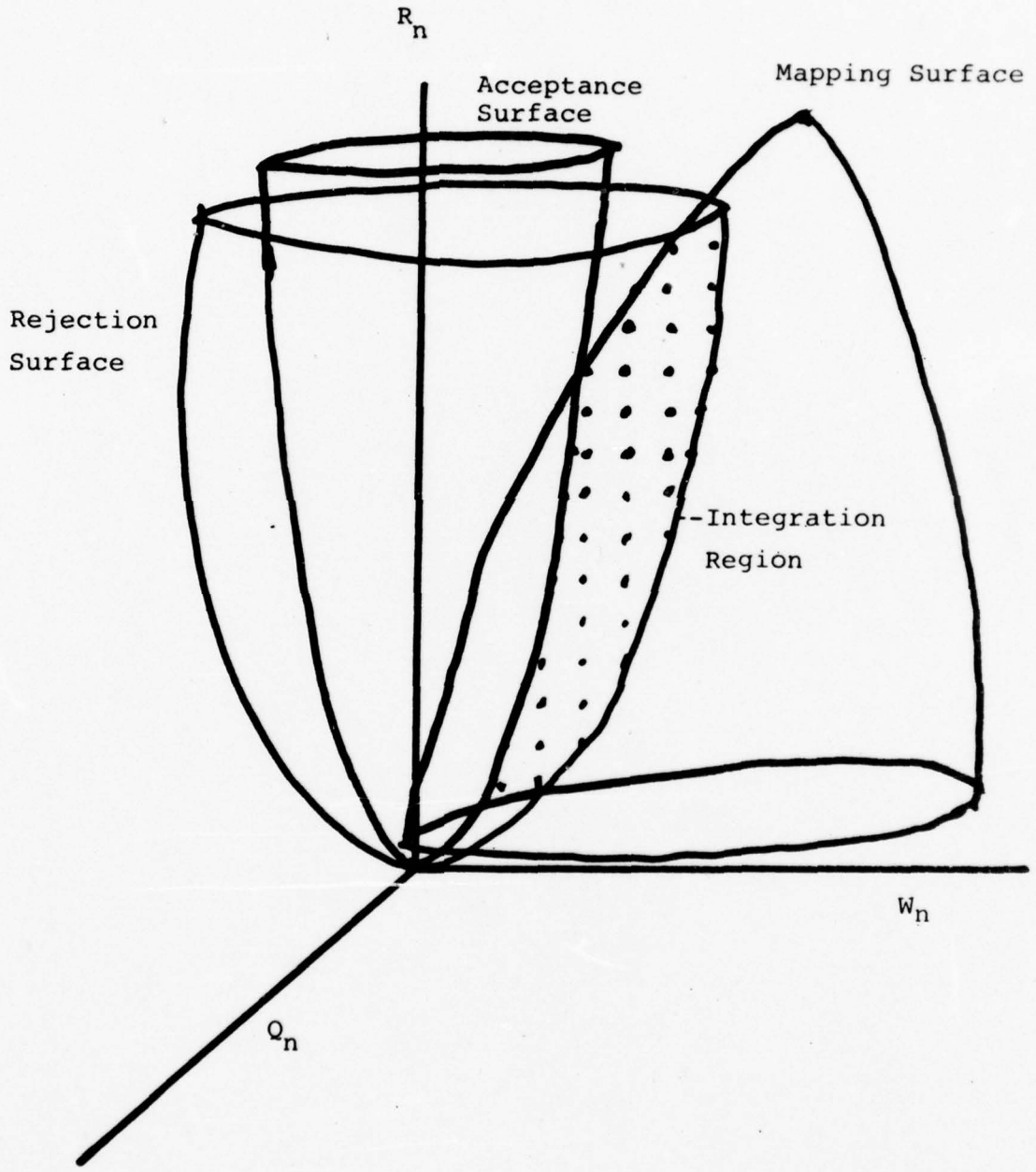
$$c = R_{n_1} + (a-W_{n_1})^2 + (b-Q_{n_1})^2 : \quad P$$

The surface P is an inverted elliptic paraboloid.

The intersection of the continuation region C_{n_1}, with the mapping surface P, determines the integration region for equation (2.3.8). This region is shown in Figure 2, and depends upon the point (a,b,c) as well as the regions V_A and V_R.

FIGURE 2

THE DIRECT METHOD INTEGRATION REGION



If this region is projected onto the W_{n_1}, Q_{n_1} axes one obtains the set of all W_{n_1}, Q_{n_1} points for which W_{n_1}, Q_{n_1}, R_{n_1} are contained on both the continuation surface and the mapping surface. Let this set be denoted by H;

$$H: \{W_{n_1}, Q_{n_1}\} \text{ s.t. } (W_{n_1}, Q_{n_1}, R_{n_1}) \in C_{n_1} \text{ and P.}$$

The integration region for U and Z, for a given point $W_{n_2} = a$, $Q_{n_2} = b$, $R_{n_2} = c$ will consist of all points U, Z such that the doublet $W = a - Z$, $Q = b - U$ is contained in the set H. Let this set be denoted by G,

$$G: \{Z, U\} \text{ such that } (a-Z, b-U) \in H.$$

Since there is a one-to-one relationship between the sets G and H, the limits U_U, U_L, Z_U, A_L can be found by inspection of the set H.

An analytic expression for H can be found by projecting $B_A \cap P$ and $B_R \cap P$ onto the W_{n_1}, Q_{n_1} axes. Since

$$B_R: \quad R_{n_1} = C_R W_{n_1}^2 + C_R Q_{n_1}^2 - 2P_R W_{n_1} Q_n$$

$$P: \quad R_{n_1} = C - (W_{n_1}-a)^2 - (Q_{n_1}-b)^2$$

the projection of $B_R \cap P$ onto the Q_{n_1}, W_{n_1} axes is obtained

by substitution, yielding the curve RE:

$$(C_R+1)\,W_{n_1}^2 + (C_R+1)\,Q_{n_1}^2 - 2aW_{n_1} - 2bQ_{n_1} - 2P_R W_{n_1} Q_{n_1} = c - a^2 - b^2$$

Similarly the projection of $B_A \cap P$ onto the Q_{n_1}, W_{n_1} axes yields the curve AE:

$$(C_A+1)\,W_{n_1}^2 + (C_A+1)\,Q_{n_1}^2 - 2aW_{n_1} - 2bQ_{n_1} - 2P_A W_{n_1} Q_{n_1} = c - a^2 - b^2$$

Both RE and AE are equations of an ellipse; and since the coefficients of $W_{n_1}^2$ and $Q_{n_1}^2$ are equal the axes of the ellipse are rotated 45°. Thus, the set H consists of all points which are inside RE and outside AE. Figure 3 shows the integration region for a particular case. Many such integration regions can arise depending upon the values of a,b,c,V_A,V_R. However, the region will always be one of the following:*

I. A point not possible at step (n+1).

This consists of all points $(W_{n+1}, Q_{n+1}, R_{n+1})$ such that $R_{n+1} - \frac{W_{n+1}^2}{n+1} - \frac{Q_{n+1}^2}{n+1} \leq 0$, which means $f(W_{n+1}, Q_{n+1}, R_{n+1}) = 0$. All future discussions about integration regions will pertain to all points possible at step (n+1).

*For examining the types of integration regions that can arise, the following less cumbersome notation will be used: $n = n_1$.

$$RE': \quad \left(\frac{n+1}{n}\right)\left[Q_n - \frac{\sqrt{2}(a+b)n}{2(n+1)}\right]^2 + \left(\frac{V_R+nV_R+1}{nV_R}\right)\left[W_n' - \frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+2}\right]^2$$

$$= \left\{C-a^2-b^2 + \frac{n(a+b)^2}{2(n+1)} + \frac{nV_R(a-b)^2}{2(V_R+nV_R+1)}\right\}$$

Note that RE will only be defined if the following inequality is satisfied

$$C-a^2-b^2 + \frac{n(a+b)^2}{2(n+1)} + \frac{nV_R(a-b)^2}{2(V_R+nV_R+1)} > 0$$

or

$$C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)} > 0$$

Whenever this inequality is not satisfied, B_R never intersects P. This means that none of the points that can be mapped into a, b, c lie in the continuation region, resulting in $f(a,b,c) = 0$.

RE' is an ellipse with center at

$$Q_n' = \frac{\sqrt{2}(a+b)n}{2(n+1)}$$

$$W_n' + \frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+1)}$$

II. Case when only a decision to reject H_0 is possible at stage n.

When V_A is a number less than zero it is not possible to accept H_0 at stage n, since the left hand side of equation (2.3.9) can never be less than zero. Assuming $V_R<\infty$, the only decision that can be made at stage n, is the decision to reject H_0.

If no decision could be made at stage n, $V_R=\infty$ and $V_A<0$, and as previously discussed, the set H consists of all W_n, Q_n inside the following circle, RE_∞:

$$\left(W_n-\frac{na}{n+1}\right)^2 + \left(Q_n-\frac{nb}{n+1}\right)^2 = \frac{n}{n+1}\left[C-\frac{a^2}{n+1} - \frac{b^2}{n+1}\right]$$

This is the set of all W_n, Q_n coordinates of all points W_n, Q_n, R_n which can be mapped into the point ($W_{n+1} = a$, $Q_{n+1} = b$, $R_{n+1} = c$) and which satisfy

$$0 < \frac{\left[W_n - Q_n\right]^2}{2\left[nR_n - W_n^2 - Q_n^2\right]} < \infty \quad .$$

Once the W_n, Q_n limits of the set H are obtained, the U, Z limits are obtained by the following relationship between the sets H and G

$$U = b - Q_n$$

$$Z = a - W_n \quad .$$

For the case when no decision can be made at stage n, these limits become those given by equations (2.3.6) and (2.3.7).

Whenever a decision to reject H_0 at stage n is possible, a set of points, W_n^*, Q_n^*, R_n^*, in W_n, Q_n, R_n space exist such that

$$V(W_n^*, Q_n^*, R_n^*) = \frac{\left[W_n^* - Q_n^*\right]^2}{2\left[nR_n^* - W_n^{*2} - Q_n^{*2}\right]} \geq V_R < \infty .$$

Since these points are not included in the set H, the set H now consists of all W_n, Q_n inside the following ellipse, RE:

$$(C_R+1)W_n^2 + (C_R+1)Q_n^2 - 2aW_n - 2bQ_n - 2P_R W_n Q_n = c - a^2 - b^2$$

To compare the two curves RE_∞ and RE consider the following rotated coordinate system:

$$Q_n' = \frac{\sqrt{2}}{2}\left[Q_n + W_n\right]$$
$$W_n' = \frac{\sqrt{2}}{2}\left[-Q_n + W_n\right]$$

The curves in this new coordinate system become

$$RE_\infty': \left[Q_n' - \frac{\sqrt{2}\,n(a+b)}{2(n+1)}\right]^2 + \left[W_n' - \frac{\sqrt{2}\,n(a-b)}{2(n+1)}\right]^2 = \frac{n}{n+1}\left(C - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right)$$

and

$$RE': \left(\frac{n+1}{n}\right)\left[Q_n' - \frac{2(a+b)n}{2(n+1)}\right]^2 + \left(\frac{V_R+nV_R+1}{nV_R}\right)\left[W_n' - \frac{2(a-b)nV_R}{2(V_R+nV_R+2)}\right]^2$$

$$= \left\{C-a^2-b^2 + \frac{n(a+b)^2}{2(n+1)} + \frac{nV_R(a-b)^2}{2(V_R+nV_R+1)}\right\}$$

Note that RE will only be defined if the following inequality is satisfied

$$C-a^2-b^2 + \frac{n(a+b)^2}{2(n+1)} + \frac{nV_R(a-b)^2}{2(V_R+nV_R+1)} > 0$$

or

$$C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)} > 0$$

Whenever this inequality is not satisfied, B_R never intersects P. This means that none of the points that can be mapped into a, b, c lie in the continuation region, resulting in $f(a,b,c) = 0$.

RE' is an ellipse with center at

$$Q_n' = \frac{\sqrt{2}(a+b)n}{2(n+1)}$$

$$W_n' = \frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+1)}$$

and minor axis along the W_n' axis. The circle RE_∞' contains the ellipse RE'. This can be seen by substituting the ellipse end points into the equation of the circle. Consider first the end points given by

$$W_n' = \frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+1)}$$

$$Q_n' = \frac{\sqrt{2}(a+b)n}{n+1} \pm \sqrt{\frac{n}{n+1} C\left\{- \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1}\right\}}$$

Substituting these points into RE_∞' yields

$$\left[\frac{n}{n+1}\left\{C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V +nV +1)}\right\}\right] + \frac{n^2(a-b)^2}{2(n+1)^2(V +nV +1)^2}$$

$$= \frac{n}{n+1}\left\{C - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right\}$$

which simplifies to

$$\frac{n^2(a-b)^2}{2(n+1)^2(V_R+nV_R+1)}\left[\frac{1}{(V_R+nV_R+1)} - 1\right]$$

Since this quantity will always be less than or equal to zero, this set of end points will be contained in RE_∞'.

Next consider the set of end points given by

$$W_n' = \frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+1)} \pm \sqrt{\frac{nV_R}{(V_R+nV_R+1}\left\{C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)}\right\}}$$

$$Q_n' = \frac{\sqrt{2}(a+b)n}{2(n+1)}$$

Substituting this into RE_∞' yields the following expression

$$\left\{-G - \frac{H(a-b)^2}{2} \mp \sqrt{2}(a-b)\sqrt{HG}\right\}\left[\frac{n}{(n+1)(V_R+nV_R+1}\right]$$

where

$$H = \frac{nV_R}{V_R+nV_R+1}$$

and

$$G = C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 H}{2}$$

The above expression may be rearranged to yield

$$\left[\sqrt{G} \pm (a-b)\sqrt{H}\ \frac{\sqrt{2}}{2}\right]^2 \left[\frac{-n}{(n+1)(V+nV+1)}\right]$$

Since this quantity will always be zero or negative, this set of end points will also be contained in RE_∞'.

Since a decision to reject H_0 can be made at stage n, the integration region is reduced to the set of all points contained in ellipse RE, as shown in Figure 3. To determine the U integration limits on the integral (2.3.8) first requires finding the Q_n limits of RE.

Letting Q_{n_U} be the maximum value of Q_n and Q_{n_L} the minimum value of Q_n on this ellipse, the integration limits for U are given by

$$U_L = b - Q_{nu}$$

$$U_U = b - Q_{n_L} \quad . \tag{2.3.11}$$

Explicit expressions for Q_{n_U} and Q_{n_L} for given a,b,c,V_R may be obtained by noting that at both points

$$\frac{dW_n}{dQ_n} = \infty \quad .$$

Therefore an expression for $\frac{dW_n}{dQ_n}$ must be found and examined to see at what points it approaches infinity.

The derivative $\frac{dW_n}{dQ_n}$ is given by

$$\frac{dW_n}{dQ_n} = \frac{P_R}{C_R+1} + \frac{\frac{1}{2}\left[\frac{2(a+P_R Q_n)}{(C_R+1)^2} + \frac{2b-2(C+1)Q_n}{(C_R+1)}\right]}{\left[\frac{(a+P_R Q_n)^2}{(C_R+1)^2} - \left(\frac{a^2+b^2-C-2bQ_n+(C_R+1)Q_n^2}{(C_R+1)}\right)\right]^{\frac{1}{2}}}$$

In order for this derivative to approach infinity the denominator must be equal to zero. Equating the numerator to zero yields,

$$\left[\frac{(a+P_R Q_n)^2}{(C_R+1)^2} - \left(\frac{a^2+b^2-C-2bQ_n+(C_R+1)Q_n^2}{(C_R+1)}\right)\right]^{\frac{1}{2}} = 0$$

and solving for Q_n yields

$$Q_n = \frac{b(C+1)+aP_R}{(C_R+1)^2-P_R^2} \pm \sqrt{\left[\frac{b(C_R+1)+aP_R}{P_R^2-C_R+1^2}\right]^2 + \left[\frac{a^2-(C+1)(a^2+b^2-c)}{P_R^2-(C+1)^2}\right]} \qquad (2.3.12)$$

The larger root will be Q_{n_U} and the smaller will be Q_{n_L}.

The limits W_{n_L} and W_{n_U} depend upon the value of Q_n. For a given value $Q' = b - U'$; $Q_{n_L} \leq Q' \leq Q_{n_U}$ the limits for W_n can be found by solving the equation of the ellipse RE, yielding

$$W_n = \frac{(a+P_R Q')}{(C_R+1)} \pm \sqrt{\left[\frac{(a+P_R Q')^2}{(C_R+1)^2} - \frac{a^2+b^2-c-2bQ'+(C_R+1)Q'^2}{C_R+1}\right]}. \tag{2.3.13}$$

Letting W_{n_L} be the smaller root and W_{n_U} the larger, the Z integration limits become

$$Z_U = a - W_{n_L} \tag{2.3.14}$$

$$Z_L = a - W_{n_U} .$$

In summary, whenever $V_A < 0$ and $V_R < \infty$ the integration limits U_L, U_U for a point $W_n = a$, $Q_n = b$, $R_n = c$, can be obtained from equation (2.3.11), where Q_{n_L} and Q_{n_U} are values obtained from equation (2.3.12). The limits Z_L and Z_U depend upon the value of U; for a given value U' the limits are obtained from equation (2.3.14) where W_{n_L} and W_{n_U} are obtained from equation (2.3.13).

III. Case when only a decision to accept H_0 is possible at stage n.

When $V_R = \infty$ a decision to reject H_0 cannot be made at stage n. If $V_R = \infty$,

$$C_R = \frac{2V_R+1}{2nV_R} = \frac{1}{n}$$

and

$$P_R = \frac{1}{2nV_R} = 0$$

so the ellipse B_R becomes:

$$R_n = \left(1+\frac{1}{n}\right) W_n^2 + \left(1+\frac{1}{n}\right) Q_n^2.$$

The projection of $B_R \cap P$ onto the W_n, Q_n axes yields, RE:

$$\left(1+\frac{1}{n}\right) W_n^2 + \left(1+\frac{1}{n}\right) Q_n^2 - 2aW_n - 2bQ_n = c - a^2 - b^2,$$

which is now the equation of a circle with center at

$$W_n = \frac{a}{1+\frac{1}{n}} = \frac{na}{n+1}$$

$$Q_n = \frac{b}{1+\frac{1}{n}} = \frac{nb}{n+1}$$

and

$$\text{radius} = \sqrt{\frac{nc}{n+1} - \frac{na^2}{(n+1)^2} - \frac{nb^2}{(n+1)^2}}.$$

The equation for AE is still given by:

$$(C_A+1)\ W_n^2 + (C_A+1)\ Q_n^2 - 2aW_n - 2bQ_n - 2P_AW_nQ_n = c - a^2 - b^2$$

which is the equation of an ellipse with center at

$$Q_n = \frac{b(C_A+1)}{(C_A-P_A+1)(C_A+P_A+1)}$$

$$W_n = \frac{a(C_A+1)}{(C_A-P_A+1)(C_A+P_A+1)}$$

In this situation the set H consists of all W_n, Q_n which lie outside the curve AE yet inside RE.

By equating the left hand sides of RE and AE one obtains the following equation:

$$W_n^2 + Q_n^2 - 2W_nQ_n = 0$$

or

$$(W_n - Q_n)^2 = 0$$

This means that the two curves, AE and RE, will intersect only at the points where $W_n = Q_n$. Substituting this into RE yields

$$2\left(1+\frac{1}{n}\right)Q_n^2 - 2(a+b)Q_n = c - a^2 - b^2$$

Solving for Q_n yields

$$\frac{(a+b) \pm \sqrt{(a+b)^2 - 2\left(1+\frac{1}{n}\right)\left(a^2+b^2-c\right)}}{2\left(1+\frac{1}{n}\right)}$$

At this point, it must be noted that there are a variety of ways in which the curves AE and RE can intersect. The above derivations have shown that when only a decision to accept H_0 is possible at stage n, the curves will intersect along the line $W_n = Q_n$. The specific points of intersection are given by the previous equation. This equation may yield zero, one, or two distinct intersection points, depending upon the value of the discriminant; i.e.

$$(a+b)^2 - 2\left(1+\frac{1}{n}\right)\left(a^2+b^2-c\right).$$

This equation reveals that the number of intersection points depends solely upon the point a, b, c.

Each of the intersection possibilities (i.e., zero, one, or two intersection points) indicates a different geometric relationship between AE and RE; which means that each results in a different U, Z integration region. Thus, to obtain the entire density (i.e. the density at all points) requires deriving the integration regions of all the possible intersection situations. Each of these possibilities will now be considered.

Whenever the following condition occurs

$$(a+b)^2 - 2\left(1+\frac{1}{n}\right)\left(a^2+b^2-c\right) \leq 0$$

the two curves RE and AE will never intersect. This indicates one of the following geometric relationships must exist:

(1) the curve RE contains AE

(2) the curve AE contains RE

(3) the curves AE and RE contain no points in common, given they don't intersect.

Situation (3) will occur only if neither curve contains the other's center. This is equivalent to satisfying the following inequalities (if they don't intersect):

$$\left[\frac{n(a-b)^2}{2V_A(n+1)^2}\right] - \left[c - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right] > 0 \qquad (2.3.15)$$

and

$$\left(\frac{(a^2+b^2)n}{n+1}\right)\left[\frac{1}{4[(n+1)V_A+1]^2} - 1\right] - \left(c - a^2 - b^2\right) > 0 \qquad (2.3.16)$$

When these inequalities are satisfied the set H consists of all W_n, Q_n contained inside RE. This is shown in Figure 4. The Q_n end points of this circle are given by:

$$\frac{nb}{n+1} \pm \sqrt{\frac{nc}{n+1} - \frac{na^2}{(n+1)^2} - \frac{nb^2}{(n+1)^2}} \qquad (2.3.17)$$

Let the smaller root be denoted by Q_{n_L} and the larger by Q_{n_U}; then the U integration limits are given by

$$U_U = b - Q_{n_L} \qquad (2.3.18)$$

$$U_L = b - Q_{n_U}$$

For a given value of U, say U^* the Z integration limits are given by

$$Z_U = a - W_{n_L} \qquad (2.3.19)$$

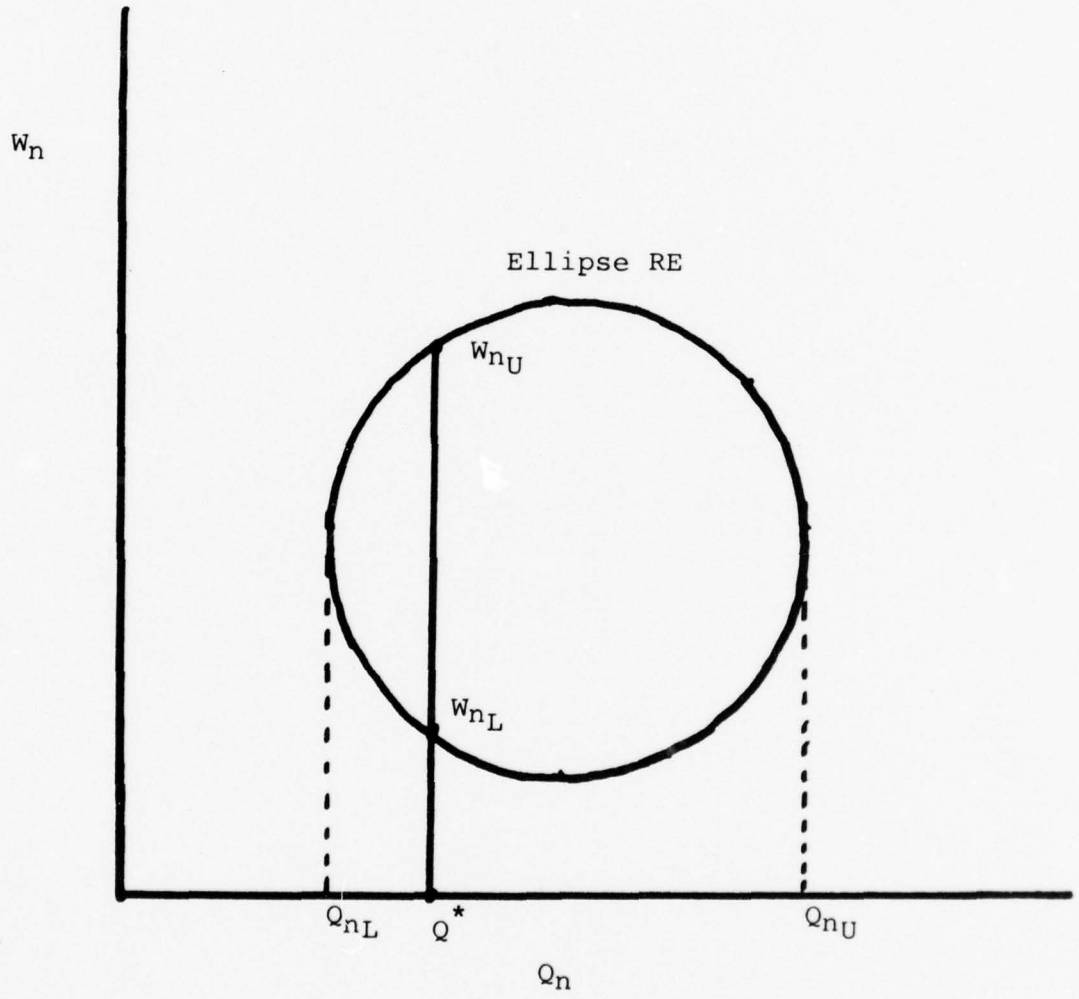
$$Z_L = a - W_{n_U}$$

Where W_{n_L} and W_{n_U} are the smallest and largest values of

$$\frac{na}{n+1} \pm \sqrt{\frac{n}{n+1}c - \frac{b^2}{n+1} - \frac{na^2}{(n+1)^2} - \frac{2bU^*}{n+1} - U^{*2}} \qquad (2.3.20)$$

FIGURE 4

Integration Region
When Neither a Decision to Accept
or Reject Can Be Made



These limits are identical to those in equations (2.3.6) and (2.3.8); which is to be expected, since situation (1) is a case where all points W_n, Q_n, R_n, which can be mapped into the point a, b, c, lie in the continuation region C_n.

Situation (2) will occur whenever the center of the circle RE is a point inside the ellipse AE; and the following point on RE

$$W_{n_c} = \sqrt{\frac{na}{n+a} + \frac{nc}{n+1} - \frac{na^2}{(n+1)^2} - \frac{nb^2}{(n+1)^2}}$$

$$Q_{n_c} = \frac{nb}{n+1} \quad , \tag{2.3.21}$$

is inside the ellipse AE. This is equivalent to satisfying the following inequalities:

$$\frac{n(a-b)^2}{2V_n(n+1)^2} - \left[c - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right] < 0 \tag{2.3.22}$$

and

$$\frac{1}{2nV_A}\left[\frac{\sqrt{n}}{(n+1)}(a-b) + \frac{1}{\sqrt{n}}\sqrt{\frac{nc}{n+1} - \frac{na^2}{(n+1)^2} - \frac{nb^2}{(n+1)^2}}\right]^2 < 0 \tag{2.3.23}$$

Since the second inequality can never be satisfied, situation (2) will never occur.

Situation (1) will occur whenever the center of the ellipse AE is a point inside RE; and the point W_{n_c}, Q_{n_c} defined in equation (2.3.21) is outside AE. The second constraint amounts to requiring the left hand side of equation (2.3.23) to be greater than zero; which will always be true. The first is equivalent to satisfying the following inequality:

$$\frac{n(a-b)^2}{2V_A(n+1)^2} - \left[c - \frac{a^2}{n+1} - \frac{b^2}{n+1}\right] < 0 \qquad (2.3.24)$$

Whenever this is true, the region of interest must be broken up into 4 subregions as shown in Figure 5. Thus the integral in equation (2.3.8) will be broken up into 4 separate integrals, so that

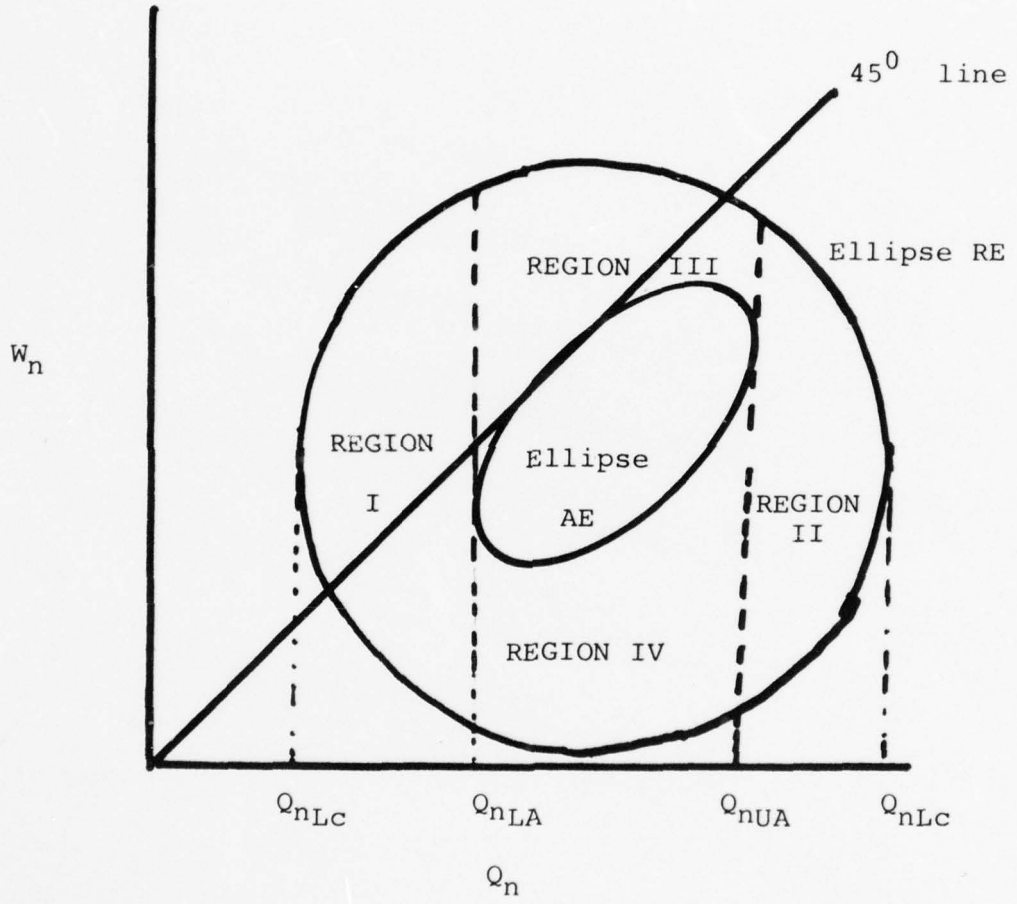
$$f_{n+1}(a,b,c) = \sum_{i=1}^{4} \left\{ \int_{U_{Li}}^{U_{Ui}} \int_{Z_{Li}}^{Z_{Ui}} f_n^P (a-Z, b-U, c-Z^2-U^2) dZ dU \right\} \qquad (2.3.25)$$

The limits U_{U_i}, U_{L_i}, Z_{U_i}, Z_{L_i} will now be obtained for each region.

For each region a range of Q_n can be found; $Q_{n_{Li}} \leq Q_n \leq Q_{n_{Ui}}$ from which the U integration limits are obtained as: $U_{L_i} = b - Q_{n_{Ui}}$ and $U_{U_i} = b - Q_{n_{Li}}$.

FIGURE 5

An Integration Region
Consisting of Four Pieces



The range of Q_n for each of the subregions is as follows:

Region I: $Q_{n_{L1}} = Q_{n_{LC}} \leq Q_n \leq Q_{n_{LE}} = Q_{n_{U1}}$

Region II: $Q_{n_{L2}} = Q_{n_{UA}} \leq Q_n \leq Q_{n_{UC}} = Q_{n_{U2}}$

Region III: $Q_{n_{L3}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U3}}$

Region IV: $Q_{n_{L4}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U4}}$

Where $Q_{n_{LC}}$ and $Q_{n_{UC}}$ are the two Q_n end points of the circle RE, or the smallest and largest values of equation (2.3.17).

$Q_{n_{LA}}$ and $Q_{n_{UA}}$ represent the Q_n end points of the ellipse AE. These are obtained by the same methods employed for Case II (equation (2.3.12)); yielding $Q_{n_{LA}}$ and $Q_{n_{UA}}$ as the smallest and largest values of

$$\frac{b(c_A+1)+aP_A}{(c_A+1)^2-P_A^2} \pm \sqrt{\left[\frac{b(c_A+1)+aP_A}{P_A^2-(c_A+1)^2}\right]^2 - \left[\frac{a^2-(c_A+1)(a^2+b^2-c)}{P_A^2-(c_A+1)^2}\right]} \qquad (2.3.27)$$

Similarly, for each region a range of W_n values can be defined: $W_{n_{Li}} \leq W_n \leq W_{n_{Ui}}$ from which the Z limits are obtained as: $Z_{L_i} = a - W_{n_{Ui}}$ and $Z_{Ui} = a - W_{n_{Li}}$.

The range of W_n for each of the subregions is as follows:

Region I: $W_{n_{L1}} = W_{n_{LC}} \leq W_n \leq W_{n_{UC}} = W_{n_{U1}}$

Region II: $W_{n_{L2}} = W_{n_{LC}} \leq W_n \leq W_{n_{UC}} = W_{n_{U2}}$

Region III: $W_{n_{L3}} = W_{n_{UA}} \leq W_n \leq W_{n_{UC}} = W_{n_{U3}}$

Region IV: $W_{n_{L4}} = W_{n_{LC}} \leq W_n \leq W_{n_{LA}} = W_{n_{U4}}$

Where $W_{n_{LC}}$ and $W_{n_{UC}}$ are W_n points on the lower and upper portion of the circle RE; and $W_{n_{LA}}$ and $W_{n_{UA}}$ the analogous points on the ellipse AE. As in the previous cases, these values will depend upon the value of Q_n, $Q_{n_{Li}} \leq Q_n \leq Q_{n_{Ui}}$, or equivalently the value of U. For a given value of U, say U^*, $W_{n_{LC}}$ and $W_{n_{UC}}$ are the smallest and largest values of equation (2.3.20). The values $W_{n_{LA}}$ and $W_{n_{UA}}$ are the smallest and largest values of the following:

$$\frac{(a+P_A Q^*)}{(C_A+1)} \pm \sqrt{\frac{a+P_A Q^{*\,2}}{C_A+1} - \frac{a^2+b^2-c-2bQ^*+(C_A+1)Q^{*2}}{C_A+1}} \qquad (2.3.28)$$

where $Q^* = b - U^*$.

The case where the curves RE and AE intersect at only one point must also be considered. This will happen whenever

$$(a+b)^2 - 2(1+\frac{1}{n})(a^2+b^2-c) = 0.$$

If this is true the curves will intersect at the point

$$W_n = Q_n = \frac{n(a+b)}{2(n+1)} . \qquad (2.3.29)$$

Based on the previous discussion this can only occur in the following situations:

(1) the curve RE contains AE;

(2) the curves AE and RE contain no points in common, except for the point of intersection given in equation (2.3.29).

Situation (2) will occur whenever the inequalities given by equations (2.3.15) and (2.3.16) are satisfied. Since the point of intersection will be on the boundary of RE, the integration regions U_U, U_L, Z_U, and Z_L can still be obtained by equations (2.3.18) and (2.3.19).

Situation (1) will occur whenever the inequality given in equation (2.3.24) is satisfied. The integration region must now be broken up into three pieces as shown in Figure 6. This is simply a special case of Figure 5 and may be evaluated by the same methods used to evaluate equation (2.3.25).

The curves RE and AE may also intersect at two points. This will happen whenever

$$(a+b)^2 - 2(1+\frac{1}{n})(a^2+b^2-c) > 0.$$

This indicates that one of the following geometric relationships must exist:

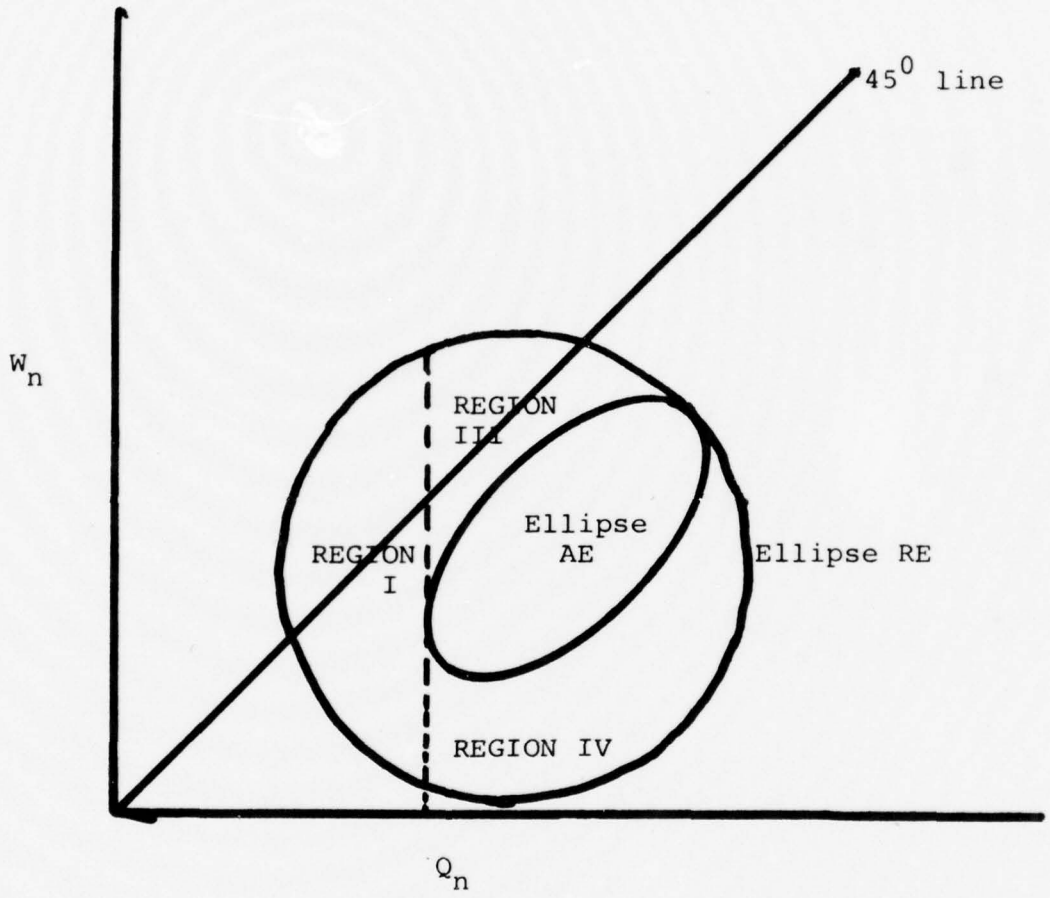
(1) The ellipse AE is contained inside the circle RE, with the Q_n end points of AE touching the circle.

(2) The intersection points fall below the axis of the ellipse AE, which is parallel to the line $W_n = Q_n$.

(3) The intersection points fall above the axis of the ellipse AE, which is parallel to the line $W_n = Q_n$.

FIGURE 6

An Integration Region
Consisting of Three Pieces



Consider the following set of rotated axes (45° rotation):

$$Q_n' = \frac{\sqrt{2}}{2}\left[Q_n + W_n\right] \qquad (2.3.29)$$

$$W_n' = \frac{\sqrt{2}}{2}\left[-Q_n + W_n\right]$$

The ellipse AE in terms of this new coordinate system becomes:

$$(C_A + 1-P_A)\ Q_n'^2 - \left[\frac{\sqrt{2}(a+b)}{2(C_A+1-P_A)}\right]^2 + (C_A+1+P_A)\left[W_n' - \frac{\sqrt{2}(a-b)}{2(C_A+1+P_A)}\right]^2$$

$$= c-a^2-b^2 + \frac{(a-b)^2}{2(C_A+1+P_A)} + \frac{(a+b)^2}{2(C_A+1-P_A)} \qquad (2.3.30)$$

Also, the line $W_n = Q_n$ becomes:

$$W_n' = 0. \qquad (2.3.31)$$

From these equations criteria for situations (1) - (3) can be established. Situation (1) will occur whenever $a = b$; situation (2) will occur whenever $a > b$; and situation (3) will occur whenever $a < b$.

Situation (1) is shown in Figure 7. The integration region must be divided into at most four subregions, as in Equation (2.3.25). The integration limits are the same as those obtained for equation (2.3.25), with the exception that one or two of the subregions may be empty.

Situation (2) is shown in Figure 8. The integration region must now be divided into three subregions. The range of Q_n for each of the subregions is as follows:

$$\begin{aligned}
&\text{Region I:} \quad Q_{n_{L1}} = Q_{n_{LC}} \leq Q_n \leq Q_{n_{LI}} = Q_{n_{U1}} \\
&\text{Region II:} \quad Q_{n_{L2}} = Q_{n_{LI}} \leq Q_n \leq Q_{n_{UI}} = Q_{n_{U2}} \\
&\text{Region III:} \quad Q_{n_{L3}} = Q_{n_{UI}} \leq Q_n \leq Q_{n_{UC}} = Q_{n_{U3}}
\end{aligned} \tag{2.3.32}$$

The quantities $Q_{n_{LC}}$ and $Q_{n_{UC}}$ are again the two Q_n end points of the circle RE, or the smallest and largest values of equation (2.3.17). $Q_{n_{LI}}$ and $Q_{n_{UI}}$ are the intersection points of the ellipse AE with the circle RE, or the smallest and largest values of the following equation:

$$\frac{(a+b) \pm \sqrt{(a+b)^2 - 2\left(1+\frac{1}{n}\right)(a^2+b^2-c)}}{2\left(1+\frac{1}{n}\right)} \tag{2.3.33}$$

FIGURE 7

Integration Region When Ellipse AE
Intersects Circle RE at Two Points
Situation 1

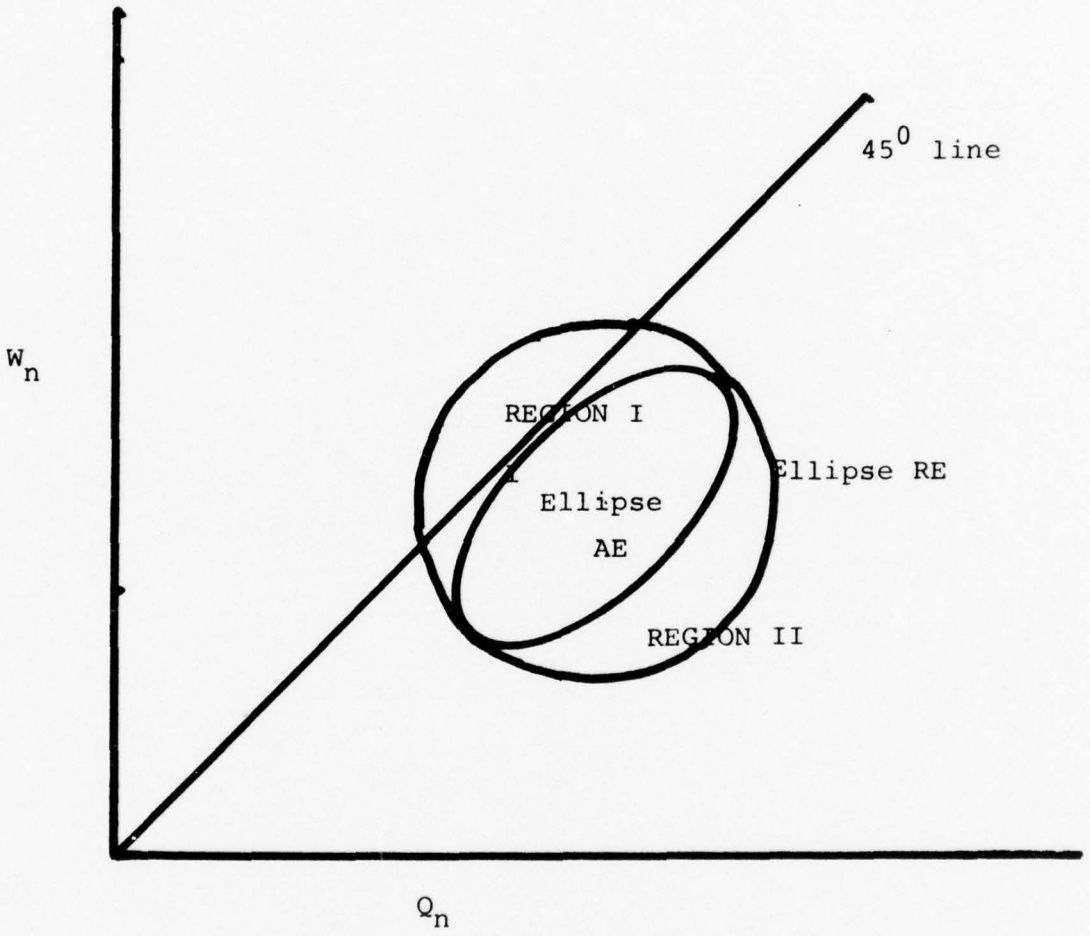
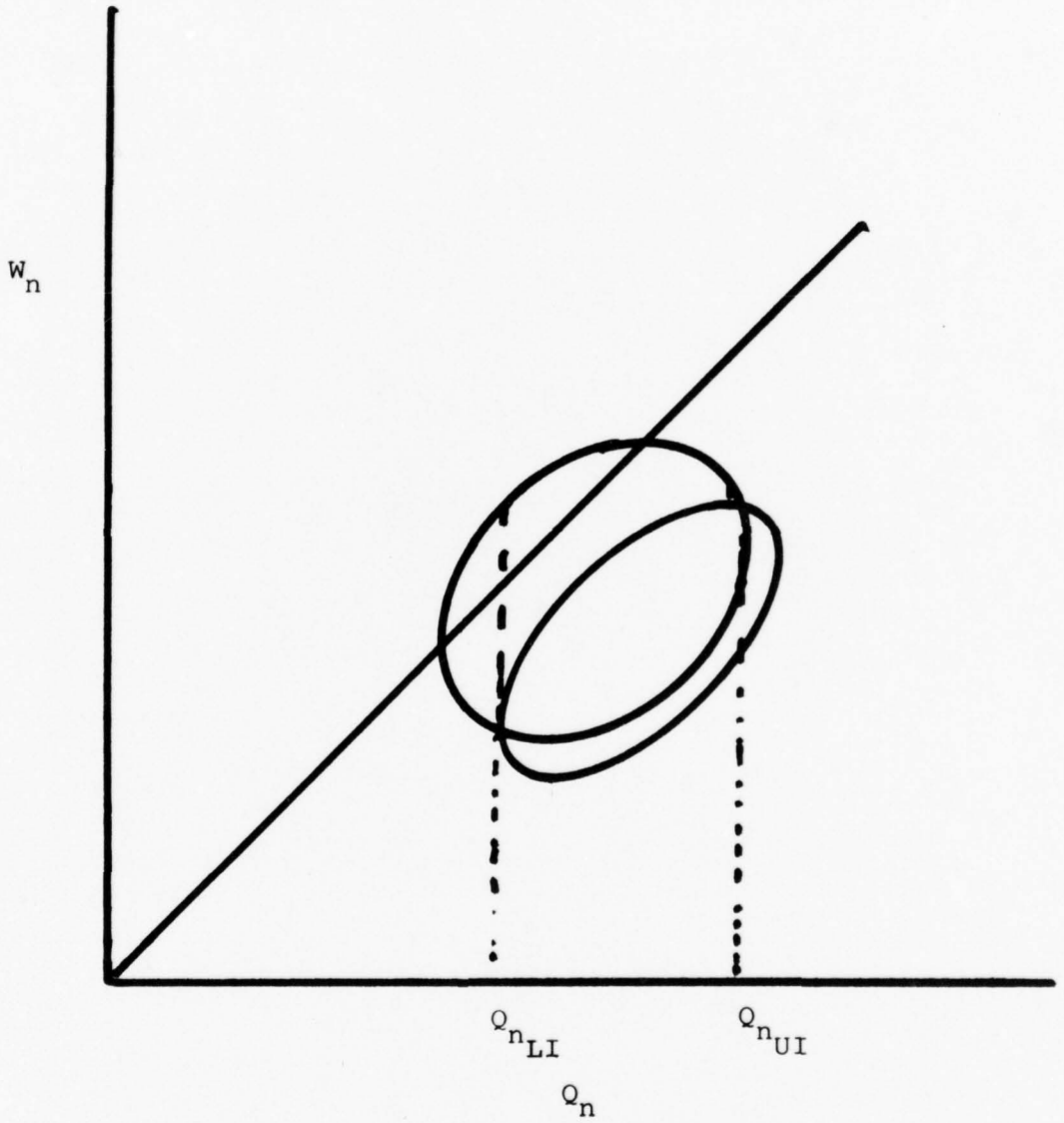


FIGURE 8

Integration Region When Ellipse AE
Intersects Circle RE at Two Points
Situation 3



The U integration limits for each region are then given by:

$$U_{Li} = b - Q_{n_{Ui}}$$

$$U_{Ui} = b - Q_{n_{Li}}$$

$$i = 1,2,3 \qquad (2.3.34)$$

The range of W_n for each of the subregions is as follows:

Region I: $W_{n_{L1}} = W_{n_{LC}} \leq W_n \leq W_{n_{UC}} = W_{n_{U1}}$

Region II: $W_{n_{L2}} = W_{n_{LC}} \leq W_n \leq W_{n_{UE}} = W_{n_{U2}}$

Region III: $W_{n_{L3}} = W_{n_{LC}} \leq W_n \leq W_{n_{UC}} = W_{n_{U3}} \qquad (2.3.35)$

For a given value of U, say U^*, $W_{n_{LC}}$ and $W_{n_{UC}}$ are the smallest and largest values of equation (2.3.20) and $W_{n_{UC}}$ is the larger value of equation (2.3.28). Thus, the Z integration limits are obtained as:

$$Z_{Li} = a - W_{n_{Ui}}$$

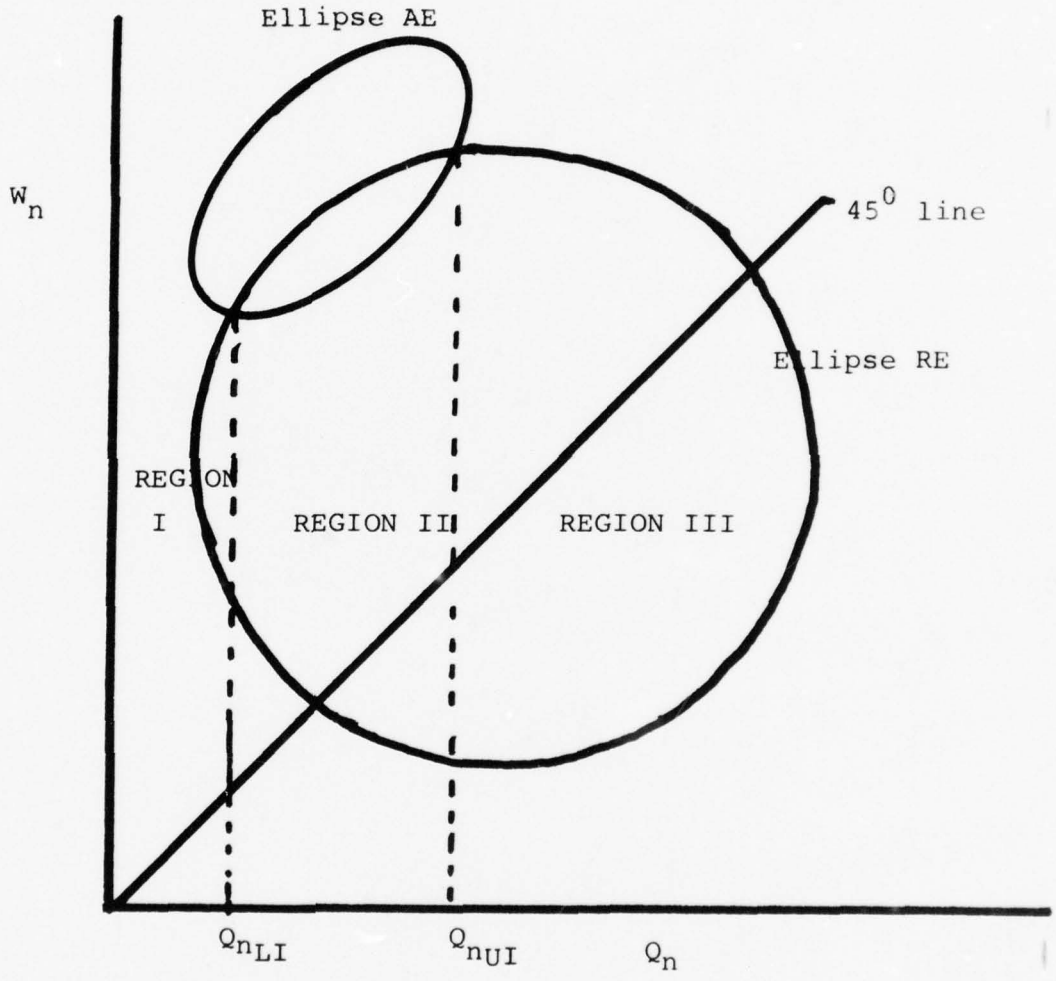
$$Z_{ui} = a - W_{n_{Li}}$$

$$i = 1,2,3 \qquad (2.3.36)$$

Situation (3) is shown in Figure 9. As in situation (2), the integration region must be broken up into three subregions. The range of Q_n for each of the subregions and the U integration limits are still given by equations (2.3.32) and

FIGURE 9

Integration Region When Ellipse AE
Intersects Circle RE at Two Points
Situation 3



(2.3.34) respectively. However, the range of W_n for each of the subregions is now:

$$\text{Region I:} \quad W_{n_{L1}} = W_{n_{LC}} \leq W_n \leq W_{n_{UC}} = W_{n_{U1}}$$

$$\text{Region II:} \quad W_{n_{L2}} = W_{n_{LC}} \leq W_n \leq W_{n_{LE}} = W_{n_{U2}}$$

$$\text{Region III:} \quad W_{n_{L3}} = W_{n_{LC}} \leq W_n \leq W_{n_{UC}} = W_{n_{U3}} \qquad (2.3.36)$$

Where $W_{n_{LE}}$ is the smaller value of equation (2.3.28). Given these values the Z integration limits are given by equation (2.3.36).

It is also possible that the ellipse AE does not exist. This will occur whenever the surface B_A does not intersect the mapping function P, or whenever the following inequality is satisfied:

$$\frac{n(a-b)^2}{2(n+1)[(n+1)V_A+1]} - C - \left[\frac{a^2}{n+1} - \frac{b^2}{n+1}\right] \geq 0 \qquad (2.3.37)$$

This is not a special case, however, because whenever this inequality is satisfied, inequalities (2.3.15) and (2.3.16) are also satisfied. Thus the integration regions are obtained from equations (2.3.17) - (2.3.20).

In summary, this section has discussed the various types of integration regions that can result when only a decision to accept H_0 is possible at stage n, criteria for determining when each of these regions is appropriate, and formulas to calculate the required U, Z limits for each of these regions.

IV. Case when either a decision to accept or reject H_0 is possible at stage n.

Whenever $0 < V_A < V_R < \infty$, both acceptance and rejection are possible at stage n. In this case, both the curves AE and RE become equations of an ellipse. In order to determine the integration region, it is necessary to know the intersection points of the two ellipses.

The intersection points are most easily found by transforming AE and RE into a coordinate system rotated 45 degrees. The equation for RE in the rotated axes becomes, RE´:

$$(C_R-P_R+1)\ Q'^{2}_{n} + (C_R+1+P_R)W'^{2}_{n} - S(2a+2b)\ Q'_n - S(2a-2b)\ W'_n = C-a^2-b^2 \qquad (2.3.30)$$

and that of AE ,

AE´:

$$(C_A-P_A+1)\ Q'^{2}_{n} + (C_A+1+P_A)W'^{2}_{n} - S(2a+2b)\ Q'_n - S(2a-2b)\ W'_n = C-a^2-b^2 \qquad (2.3.31)$$

where S is given by

$$S = \text{Sin}(45°) = \frac{\sqrt{2}}{2}$$

These equations may be further simplified as:

RE´ :

$$(C_R-P_R+1)\left\{Q_n' - \frac{S(a+b)}{(C_R-P_R+1)}\right\}^2 + (C_R-P_R+1)\left\{W_n' - \frac{S(a-b)}{(C_R+P_R+1)}\right\}^2$$

$$= c-a^2-b^2 + \frac{(a+b)^2}{2(C_R-P_R+1)} + \frac{(a-b)^2}{2(C_R+P_R+1)}$$

and

AE´ :

$$(C_A-P_A+1)\left\{Q_n' - \frac{S(a+b)}{(C_A-P_A+1)}\right\}^2 + (C_A+P_A+1)\left\{W_n' - \frac{S(a-b)}{(C_A+P_A+1)}\right\}^2$$

$$= c-a^2-b^2 + \frac{(a+b)^2}{2(C_A-P_A+1)} + \frac{(a-b)^2}{2(C_A+P_A+1)}$$

Since

$$P_A = \frac{1}{2nV_A}$$

and

$$C_A = \frac{2V_A+1}{2nV_A} ,$$

(with similar expressions for P_R and C_R), these may be substituted into the above; which yields after combining similar terms, the following expressions:

RE´ :

$$(C_R-P_R+1)\left\{Q_n' - \frac{S(a+b)}{(C_R-P_R+1)}\right\}^2 + (C_R+P_R+1)\left\{W_n' - \frac{S(a-b)}{(C_R+P_R+1)}\right\}^2$$

$$= C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)}$$

and

AE´ :

$$(C_A-P_A+1)\left\{Q_n' - \frac{S(a+b)}{(C_A-P_A+1)}\right\}^2 + (C_A+P_A+1)\left\{W_n' - \frac{S(a-b)}{(C_A+P_A+1)}\right\}^2$$

$$= C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_A+nV_A+1)}$$

Since

$$C_A-P_A+1 = C_R-P_R+1 ,$$

the above equations show that both curves will have identical Q_n' coordinates for their centers.

Solving for Q'_n in RE′ yields a solution of the following form:

$$Q'_n = K \pm D$$

where

$$K = \frac{S(a+b)}{(C_R-P_R+1)}$$

$$D = \sqrt{\frac{.5(a+b)^2}{(C_R-P_R+1)} - \left[\frac{a^2+b^2-c+(C_R+1+P_R)W'^2_n-S(2a+2b)W'_n}{C_R-P_R+1}\right]}$$

Substituting this form in the equation for AE′ yields

$$(C_A-P_A+1)(K^2+D^2)+(C_A+1+P_A)W'^2_n \qquad (2.3.32)$$

$$- S(2a+2b)K - S(2a-2b)W'_n$$

$$+ \left(\pm 2KD(C_A-P_A+1) - S(2a+2b)(\pm D)\right)$$

$$= C-a^2-b^2$$

In general an equation describing the intersection of two ellipses will be a quartic. Equation (2.3.32) is a special case, however, and can be reduced to a quadratic by noting that the following term

$$\pm D\left(2K(C_n-P_A+1) - S(2a+2b)\right)$$

$$= \pm D \left[\frac{S(2a+2b)(C_A - P_A + 1)}{(C_R - P_R + 1)} - S(2a+2b) \right]$$

is zero since

$$(C_A - P_A + 1) = (C_R - P_R + 1) = \frac{n+1}{n} .$$

Solving equation (2.3.32) for W_n' and substituting into RE' yields the following W_n', Q_n' intersection points:

$$W_n' = 0 \qquad (2.3.33)$$

$$Q_n' = \frac{1}{n+1} \left[S(a+b)n \pm \sqrt{S\left[(a+b)n\right]^2 - n(n+1)(a^2+b^2-c)} \right]$$

In terms of the original W_n, Q_n axes the intersection points become:

$$W_n = Q_n = \frac{1}{2(n+1)} \left[n(a+b) \pm \sqrt{\left[(a+b)n\right]^2 - n(n+1)(a^2+b^2-c)} \right] \qquad (2.3.34)$$

This means that the two ellipses, AE and RE, will intersect at zero, one, or two points. The sign of the discriminant of equation (2.3.34),

$$DIS = \left[(a+b)n\right]^2 - n(n+1)(a^2+b^2-c),$$

determines the number of intersection points. Whenever:

1. $DIS < 0$ (2.3.35)

AE and RE will not intersect.

2. DIS = 0 (2.3.36)

AE and RE will intersect at only one point, this point being

$$W_n = Q_n = \frac{n(a+b)}{2(n+1)}$$

3. DIS > 0 (2.3.37)

AE and RE will intersect at two points.

Consider first, the case when AE and RE do not intersect, or when equation (2.3.35) is satisfied. This indicates one of the following geometric relationships must exist:

(1) the curves AE and/or RE do not exist

(2) the curve RE contains AE

(3) the curve AE contains RE

(4) the curves AE and RE contain no points in common.

Situation (1) will occur if either of the ellipses has an imaginary radius, or whenever either of the following equations is satisfied:

$$C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R + nV_R + 1)} \leq 0 \qquad (2.3.38)$$

$$C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_A + nV_A + 1)} \leq 0 \; . \qquad (2.3.39)$$

Since $V_A \leq V_R$, inequality (2.3.39) will be satisfied whenever (2.3.38) is satisfied. If (2.3.38) is satisfied, $f(a,b,c) = 0$, since none of the points that can be mapped into a,b,c lie in the continuation region.

If inequality (2.3.39) is satisfied and (2.3.38) is not, the point a,b,c is located such that the mapping function P intersects the rejection surface but never intersects the acceptance curve. This reduces to a case previously discussed, the case when only a decision to reject H_0 is possible, and the integration regions are given by equations (2.3.11) through (2.3.14).

Assuming neither inequality (2.3.38) or (2.3.39) is satisfied, situation (4) will occur when neither curve contains the other's center, or when the following inequalities are satisfied:

$$\left(\frac{V_R+nV_R+1}{nV_R}\right)\left(\frac{n^2(a-b)^2}{2}\right)\left[\frac{V_R}{(V_R+nV_R+1)} - \frac{V_A}{(V_A+nV_A+1)}\right]^2$$

$$- \left\{C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_R+nV_R+1)}\right\} > 0 \qquad (2.3.40)$$

and

$$\left(\frac{V_A+nV_A+1}{nV_A}\right)\left(\frac{n^2(a-b)^2}{2}\right)\left[\frac{V_R}{(V_R+nV_R+1)} - \frac{V_A}{(V_A+nV_A+1)}\right]^2$$

$$- \left\{C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_A+nV_A+1)}\right\} > 0 \qquad (2.3.41)$$

As it is necessary for both inequalities to hold, and inequality (2.3.41) implies inequality (2.3.40), it is only necessary to examine the former. In other words, situation (4) will occur whenever RE and AE do not intersect, and RE does not contain AE's center point.

The set H consists of all points W_n, Q_n contained inside the ellipse RE. The integration region in this case becomes identical to that required for the case when only rejection is possible and can be evaluated using equations (2.3.11) - (2.3.14).

Situation (3) will occur whenever the four end points of the ellipse RE´ are all points inside AE.´ First,

consider the RE´ end points along the Q_n´ axis. When this is substituted into AE´, the following inequality must hold for situation (3) to occur:

$$\left(\frac{n(a-b)^2}{2}\right)\left(\frac{V_R}{(V_R+nV_R+1)} - \frac{V_A}{(V_A+nV_A+1)}\right)$$

$$+ \left(\frac{V_A+nV_A+1}{nV_A}\right)\left(\frac{(a-b)^2n^2}{2}\right)\left(\frac{V_R}{(V_R+nV_R+1)} - \frac{V_A}{(V_R+nV_A+1)}\right)^2 < 0 .$$

Since $V_A \leq V_R$, this inequality can not be satisfied; and thus situation (3) can never occur.

Having shown that situation (3) cannot occur, situation (2) will occur whenever both the ellipses RE and AE exist (neither inequality (2.3.38) nor (2.3.31) is satisfied), and inequality (2.3.41) is not satisfied. In this case the integral given in equation (2.3.8) must be broken up into four separate pieces, similar to that given in equation (2.3.25). The limits U_{Ui}, U_{Li}, Z_{Ui} and Z_{Li}: i = 1,...,4 must be determined for each region.

For each region a range of Q_n can be found; $Q_{n_{Li}} \leq Q_n \leq Q_{n_{Ui}}$ from which the U integration limits are obtained as:

$$U_{Li} = b - Q_{n_{Ui}}$$

$$U_{Ui} = b - Q_{n_{Li}} .$$

The range of Q_n for each of the pieces is as follows:

$$\text{Region I:} \quad Q_{n_{L1}} = Q_{n_{LR}} \leq Q_n \leq Q_{n_{LA}} = Q_{n_{U1}}$$

$$\text{Region II:} \quad Q_{n_{L2}} = Q_{n_{UA}} \leq Q_n \leq Q_{n_{UR}} = Q_{n_{U2}}$$

$$\text{Region III:} \quad Q_{n_{L3}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U3}}$$

$$\text{Region IV:} \quad Q_{n_{L4}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U4}} \qquad (2.3.42)$$

Where $Q_{n_{LR}}$ and $Q_{n_{UA}}$ are the minimum and maximum Q_n coordinates on the ellipse RE; and $Q_{n_{LA}}$ and $Q_{n_{UA}}$ are the analogous quantities on the ellipse AE.

$Q_{n_{LR}}$ and $Q_{n_{UR}}$ have previously been derived as the smallest and largest values of equation (2.3.12). A similar expression derived for RE, yields $Q_{n_{LA}}$ and $Q_{n_{UA}}$ as the smallest and largest values of

$$\left[\frac{b(C_A+1) + aP_A}{(C_A+1)^2 - P_A^2}\right] \pm \sqrt{\left[\frac{b(C_A+1) + aP_A}{P_A^2 - (C_A+1)^2}\right]^2 - \left[\frac{a^2-(C_A+1)(a^2+b^2-c)}{P_A^2 - (C_A+1)^2}\right]} \qquad (2.3.43)$$

The range of W_n for each piece depends upon the value of Q_n (or equivalently U). For a given value of U_i, say U*, $U^* = b - Q_{n_i}^*$; $Q_{n_{Li}} \leq Q_{n_i}^* \leq Q_{n_{Ui}}$ a range of W_n values can be defined; $W_{n_{Li}} \leq W_n \leq W_{n_{ui}}$, from which the Z limits are obtained as:

$$Z_{L_i} = a - W_{n_{Ui}}$$

$$Z_{U_i} = a - W_{n_{Li}} \quad .$$

The range of W_n for each of the pieces is as follows:

$$\text{Region I:} \quad W_{n_{L1}} = W_{n_{LR}} \leq W_n \leq W_{n_{UR}} = W_{n_{U1}}$$

$$\text{Region II:} \quad W_{n_{L2}} = W_{n_{LR}} \leq W_n \leq W_{n_{UR}} = W_{n_{U2}}$$

$$\text{Region III:} \quad W_{n_{L3}} = W_{n_{UA}} \leq W_n \leq W_{n_{UR}} = W_{n_{U3}}$$

$$\text{Region IV:} \quad W_{n_{L4}} = W_{n_{LR}} \leq W_n \leq W_{n_{LA}} = W_{n_{U4}}$$

(2.3.44) .

$W_{n_{LR}}$ and $W_{n_{UR}}$ are the upper and lower points on the ellipse **RE**, for a given value of $U^* = b - Q_n^*$. These have been derived previously as the smallest and largest values of equation (2.3.13). $W_{n_{LA}}$ and $W_{n_{UA}}$ are the analogous points on the ellipse AE, and are obtained as the smallest and largest value of:

$$\left[\frac{(a+P_A Q^*)}{(C_A+1)}\right] \pm \sqrt{\left[\frac{a+P_A Q^*}{C_A+1}\right]^2 - \left[\frac{a^2+b^2-C-2bQ^*+(C_A+1)Q^{*2}}{(C_A+1)}\right]}$$

(2.3.45) .

Next consider the case when AE and RE intersect at only one point, which will occur whenever equation (2.3.36) is satisfied. Based on the previous discussion this can only occur in the following situations:

(1) either one or both of the curves AE and RE do not exist

(2) the curve RE contains AE

(3) the curves AE and RE contain no points in common, except for the point of intersection.

Situation (1) can never occur if equations (2.3.36) or (2.3.37) hold. This can be shown as follows:

If

$$(a+b)^2 - \left(2\,\frac{n+1}{n}\right)\left(a^2+b^2-c\right) \geq 0$$

then

$$C = a^2+b^2 - \frac{(a+b)^2}{2(n+1)} + \frac{S_1}{2(n+1)}$$

S_1 being a quantity greater than or equal to zero. Substituting this result into the equation of the radius of the ellipse AE and simplifying yields:

$$\text{Radius AE} = \frac{(a-b)^2}{2(C_A+1+P_A)} + \frac{S_1}{2(n+1)} \quad .$$

Since this quantity will always be greater than equal to zero, the ellipse AE will always exist. The previous section also showed that a sufficient condition for RE to exist was the existence of AE. Hence, intersection of the ellipses AE and RE is a sufficient condition for their existence.

Situation (3) will occur whenever the inequality given in equation (2.3.11) is satisfied. Since the point of intersection will be on the boundary of RE, the integration regions U_L, U_L, Z_U, and Z_L can still be obtained by equations (2.3.11) and (2.3.14).

Similarly situation (2) occurs whenever inequality (2.3.41) is not satisfied, and requires the integration to be broken up into four pieces. The integration limits in each of these pieces may still be obtained by equations (2.3.42) through (2.3.45).

The curves AE and RE will intersect at two points whenever inequality (2.3.37) holds. In general, two ellipses can intersect at two points in many ways. However, consider the equations in the rotated axes coordinated system:

RE´:

$$\left(\frac{n+1}{n}\right)\left\{Q_n' - \left[\frac{\sqrt{2}(a+b)n}{2(n+1)}\right]\right\}^2 + \left(\frac{V_R+nV_R+1}{nV_R}\right)\left\{W_n' - \left[\frac{\sqrt{2}(a-b)nV_R}{2(V_R+nV_R+1)}\right]\right\}^2$$

$$= C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a+b)^2 n}{2(n+1)(V_R+nV_R+1)}$$

and

AE´:

$$\left(\frac{n+1}{n}\right)\left\{Q_n' - \left[\frac{\sqrt{2}(a+b)n}{2(n+1)}\right]\right\}^2 + \left(\frac{V_A+nV_A+1}{nV_A}\right)\left\{W_n' - \left[\frac{\sqrt{2}(a-b)nV_A}{2(V_A+nV_A+1)}\right]\right\}^2$$

$$= C - \frac{a^2}{n+1} - \frac{b^2}{n+1} - \frac{(a-b)^2 n}{2(n+1)(V_A+nV_A+1)}$$

From these equations the following relationships may be noted:

(a) the curves RE' and AE' will have parallel major axes (i.e., parallel to the line $W_n' = 0$).

(b) the major axis of RE' will be greater than or equal to the major axis of AE'.

(c) the major axes of RE' and AE' will always lie on the same side of the line $W_n' = 0$.

(d) the two curves will have the same center point and equal major axes whenever $a = b$.

(e) since

$$Q_n' = \frac{\sqrt{2}}{2}\left[Q_n + W_n\right]$$

$$W_n' = \frac{\sqrt{2}}{2}\left[-Q_n + W_n\right],$$

if the curves intersect, the intersection points will lie along the line $W_n = Q_n$ or $W_n' = 0$.

Given these relationships one can conclude that whenever the curves intersect at two points, one of the following geometric situations must exist:

(1) the ellipse RE' circumscribes the ellipse AE'.

(2) the major axes of the ellipses lie above the line $W_n' = 0$.

(3) the major axes of the ellipses lie below the line $Wn' = 0$.

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AN EXACT TEST FOR THE SEQUENTIAL ANALYSIS OF VARIANCE.(U)
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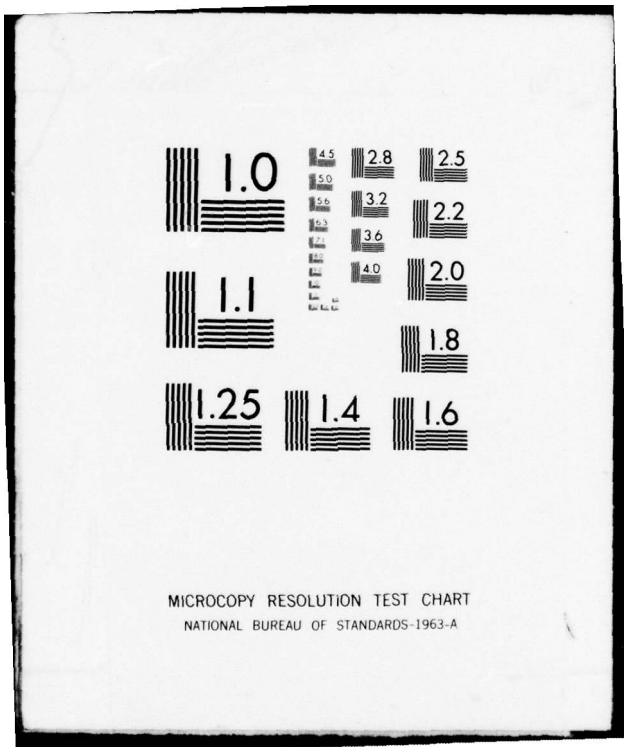
1.0
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5.0
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6.3
7.1
8.0
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Situation (1) will occur whenever $a = b$, and requires the integral of equation (2.3.8) to be broken up into two pieces. These two pieces may be described by W_n, Q_n regions identical to those of Regions III and IV given in (2.3.42) and (2.3.44). Thus the integration limits U_{Li}, U_{Ui}, Z_{Li}, and Z_{Ui} may be obtained from equations (2.3.43) - (2.3.45).

Situation (2) will occur whenever $a > b$, and requires the integral of equation (2.3.8) to be broken up into five pieces, as shown in Figure 10. The range of Q_n for each of the subregions is as follows:

$$\text{Region I: } Q_{n_{L1}} = Q_{n_{LR}} \leq Q_n \leq Q_{n_{LA}} = Q_{n_{U1}}$$

$$\text{Region II: } Q_{n_{L2}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U2}}$$

$$\text{Region III: } Q_{n_{L3}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{LI}} = Q_{n_{U3}}$$

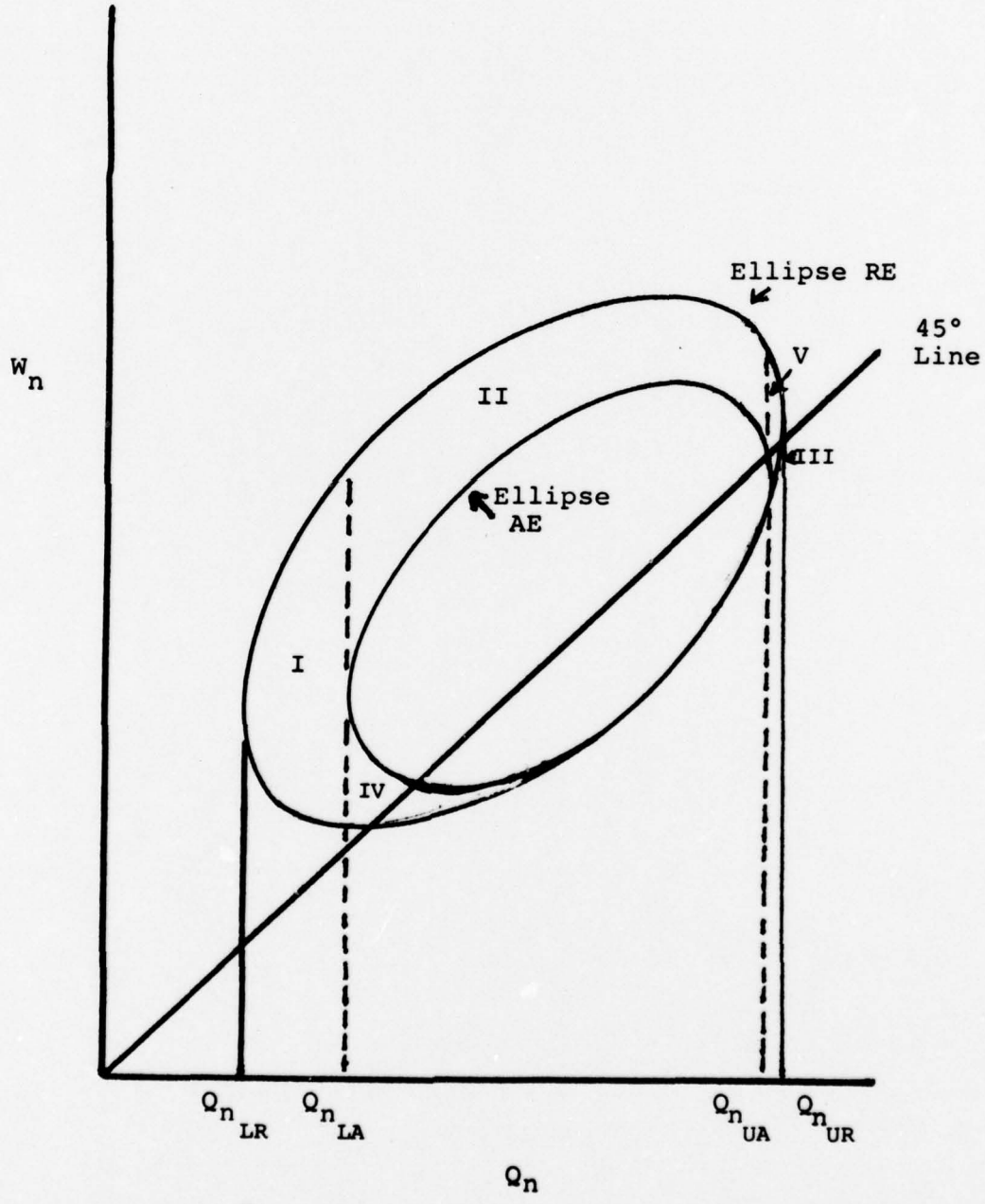
$$\text{Region IV: } Q_{n_{L4}} = Q_{n_{UI}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U4}}$$

$$\text{Region V: } Q_{n_{L5}} = Q_{n_{UA}} \leq Q_n \leq Q_{n_{UR}} = Q_{n_{U5}} \qquad (2.3.46)\ .$$

The quantities $Q_{n_{LA}}$ and $Q_{n_{UA}}$ have previously been defined and may be obtained as the minimum and maximum of (2.3.43). Similarly $Q_{n_{LR}}$ and $Q_{n_{UR}}$ are the minimum and maximum of (2.3.12). $Q_{n_{LI}}$ and $Q_{n_{UI}}$ represent the two intersection points of AE and RE, and are defined as:

FIGURE 10

Integration Region When Both a Decision
to Accept and Reject is Possible
Situation 2



$$Q_{n_{LI}} = \min \left\{R_1, R_2\right\}$$

$$Q_{n_{UI}} = \max \left\{R_1, R_2\right\}$$

where

$$R_1 = \frac{(a+b)n + n\sqrt{(a+b)^2 - 2\left(\frac{n+1}{n}\right)\left(a^2+b^2-c\right)}}{2(n+1)}$$

$$R_2 = \frac{n(a+b) - n\sqrt{(a+b)^2 - 2\left(\frac{n+1}{n}\right)\left(a^2+b^2-c\right)}}{2(n+1)} \qquad (2.3.47).$$

The U integration limits for each piece are again obtained as:

$$U_{Ui} = b - Q_{n_{Li}}$$

$$U_{Li} = b - Q_{n_{Ui}}$$

Similarly for a given value of U, say $U^* = b - Q^*$, the range of W_n for each region is given by:

Region I: $W_{n_{L1}} = W_{n_{LR}} \leq W_n \leq W_{n_{UR}} = W_{n_{U1}}$

Region II: $W_{n_{L2}} = W_{n_{UA}} \leq W_n \leq W_{n_{UR}} = W_{n_{U2}}$

Region III: $W_{n_{L3}} = W_{n_{LR}} \leq W_n \leq W_{n_{LA}} = W_{n_{U3}}$

Region IV: $W_{n_{L4}} = W_{n_{LR}} \leq W_n \leq W_{n_{LA}} = W_{n_{U4}}$

Region V: $W_{n_{L5}} = W_{n_{LR}} \leq W_n \leq W_{n_{UR}} = W_{n_{U5}}$

$W_{n_{LR}}$ and $W_{n_{UR}}$ being the maximum and minimim of (2.3.45), and $W_{n_{LA}}$ and $W_{n_{UA}}$ the same for (2.3.13).

The Z limits are obtained for each region as:

$$Z_{Ui} = a - W_{n_{Li}}$$

$$Z_{Li} = a - W_{n_{Ui}}$$

Situation (3) results whenever $a < b$, and again requires that equation (2.3.8) be split up into five separate integrals, as shown in Figure 11. In this case the range of Q_n for each of the subregions is as follows:

Region I: $Q_{n_{L1}} = Q_{n_{LR}} \leq Q_n \leq Q_{n_{LA}} = Q_{n_{U1}}$

Region II: $Q_{n_{L2}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U2}}$

Region III: $Q_{n_{L3}} = Q_{n_{LA}} \leq Q_n \leq Q_{n_{LI}} = Q_{n_{U3}}$

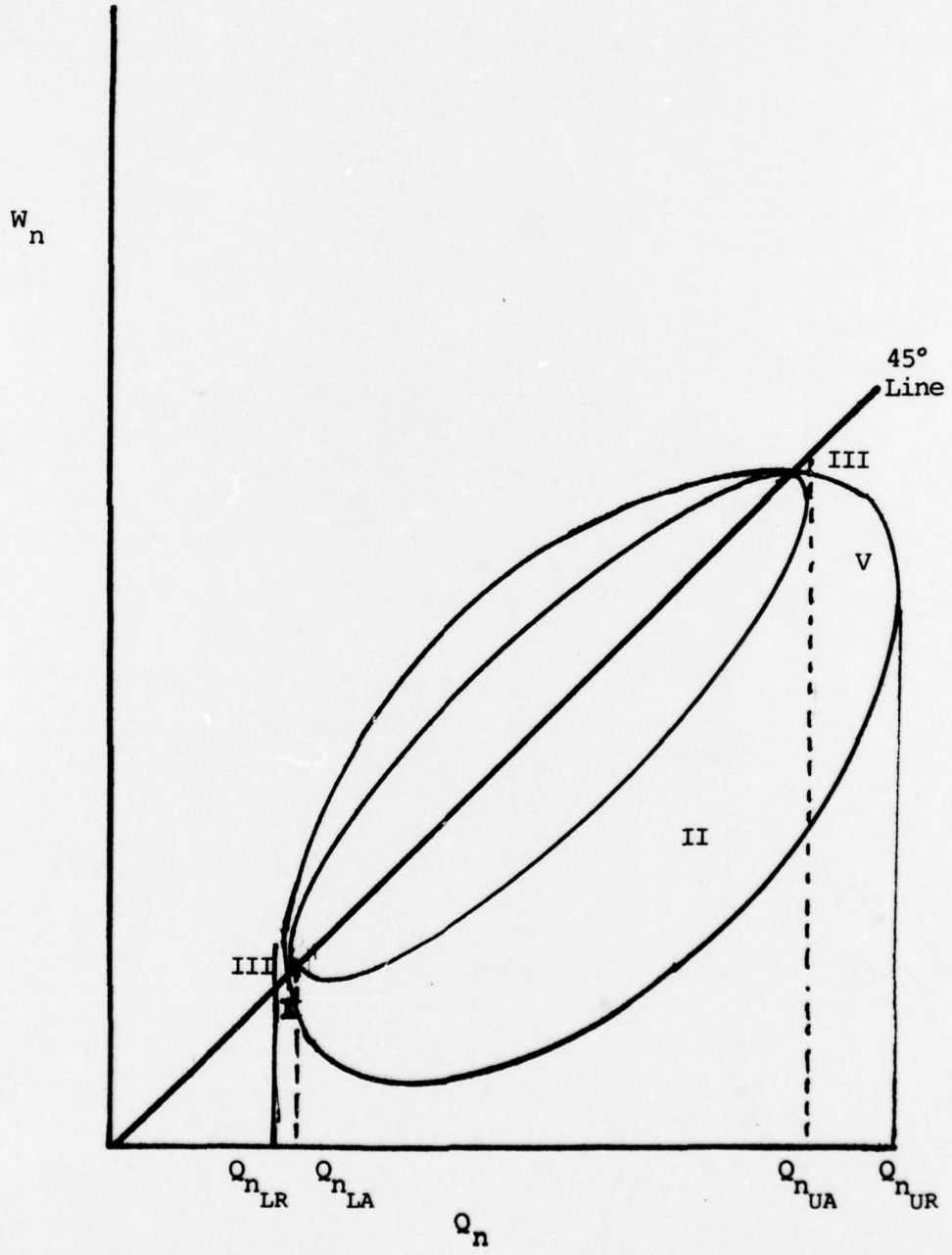
Region IV: $Q_{n_{L4}} = Q_{n_{UI}} \leq Q_n \leq Q_{n_{UA}} = Q_{n_{U4}}$

Region V: $Q_{n_{L5}} = Q_{n_{UA}} \leq Q_n \leq Q_{n_{UR}} = Q_{n_{U5}}$

(2.3.48).

FIGURE 11

Integration Region When Both a Decision to Accept and Reject is Possible Situation 3



For a given value of $U^* = b - Q^*$ the W_n range for each region is:

Region I: $W_{n_{L1}} = W_{n_{LR}} \leq W_n \leq W_{n_{UR}} = W_{n_{U1}}$

Region II: $W_{n_{L2}} = W_{n_{LR}} \leq W_n \leq W_{n_{LA}} = W_{n_{U2}}$

Region III: $W_{n_{L3}} = W_{n_{UA}} \leq W_n \leq W_{n_{UR}} = W_{n_{U3}}$

Region IV: $W_{n_{L4}} = W_{n_{UA}} \leq W_n \leq W_{n_{UR}} = W_{n_{U4}}$

Region V: $W_{n_{L5}} = W_{n_{LR}} \leq W_n \leq W_{n_{UR}} = W_{n_{U5}}$

(2.3.49).

2.4 OBTAINING THE PROBABILITIES OF ACCEPTANCE, REJECTION AND CONTINUATION

The previous section (2.3) has given methods for calculating the density $f_{n+1}(a,b,c)$, for a given point $W_{n+1} = a$, $Q_{n+1} = b$, $R_{n+1} = c$, from the density at stage n, $f_n(W_n,Q_n,R_n)$. Once this density has been obtained for all possible values of a, b, c, the probability of accepting H_0 (P_A^{n+1}), probability of rejecting H_0 as (P_R^{n+1}), and the probability of continuing (P_C^{n+1}) must be calculated. This requires integrating the three dimensional density $f_{n+1}(W_{n+1},Q_{n+1},R_{n+1})$ over all values of W_{n+1}, Q_{n+1}, R_{n+1} for which the statistic

$$V\left(W_{n+1},Q_{n+1},R_{n+1}\right) = \frac{\left[W_{n+1}-Q_{n+1}\right]^2}{2\left[(n+1)R_{n+1}-Q_{n+1}^2-W_{n+1}^2\right]}$$

is in the appropriate region (e.g., acceptance region, rejection region, or continuation region). Thus P_A^{n+1}, P_R^{n+1}, and P_C^{n+1} may be calculated as:

$$P_R^{n+1} = \iiint_{0 \le V(W,Q,R) \le V_R^{n+1}} f_{n+1}(W,Q,R)\,dW\,dQ\,dR$$

$$P_A^{n+1} = \iiint_{V_R^{n+1} \leq V(W,Q,R) \leq \infty} f_{n+1}(W,Q,R)\, dW\, dQ\, dR$$

and

$$P_C^{n+1} \iiint_{V_A^{n+1} \leq V(W,Q,R) \leq V_A^{n+1}} f_{n+1}(W,Q,R)\, dW\, dQ\, dR$$

These integrals amount to integrating $f_{n+1}(W,Q,R)$ over elliptic paraboloids, and may be reexpressed as the following iterated integrals:

$$P_A^{n+1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{R_a}^{\infty} f_{n+1}(W,Q,R)\, dR\, dW\, dQ$$

$$P_R^{n+1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{R_o}^{R_r} f_{n+1}(W,Q,R)\, dR\, dW\, dQ$$

$$P_C^{n+1} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{R_r}^{R_a} f_{n+1}(W,Q,R)\, dR\, dW\, dQ \qquad (2.4.1).$$

with

$$R_o = \frac{W^2}{n+1} + \frac{Q^2}{n+1}$$

$$R_a = \frac{[W-Q]^2}{2(n+1)V_A^{n+1}} + \frac{W^2}{n+1} + \frac{Q^2}{n+1}$$

$$R_r = \frac{[W-Q]^2}{q(n+1)V_R^{n+1}} + \frac{W^2}{n+1} + \frac{Q^2}{n+1}$$

In practice only two of the three integrals need be calculated due to the following identity:

$$P_C^{\,n} = P_A^{\,n+1} + P_R^{\,n+1} + P_C^{\,n+1} .$$

So if $P_A^{\,i}$ and $P_R^{\,i}$ are calculated at each stage i, $P_C^{\,i}$ may be obtained by subtraction,

$$P_C^{\,i} = P_C^{\,i-1} - P_A^{\,i}\ P_R^{\,i} .$$

2.5 SUMMARY OF THE DIRECT METHOD FOR A k=2 SANOVA TEST

The purpose of this section is to summarize the procedure for obtaining the OC and ASN curves for a k=2 SANOVA test.

First, a test of this type requires specification of the following quantities:

(1) The null hypothesis value, λ_0.

(2) The alternative hypothesis value, λ_1.

(3) A truncation point, m_0.

(4) A set of regions: V_A^i, V_R^i, $i = 2,\ldots, m_0$, such that at any stage N

These regions are to be compared with the statistic, V_n, of equation (2.3.1), such that at any stage n,

(a) H_0 is accepted if $V_n \leq V_A^n$

(b) H_1 is accepted if $V_n \geq V_R^n$.

(5) Values of α and β (needed only if the regions are to be modified Wald regions).

Second, the first step at which a decision can be made, say n_1, $2 \leq n_1 \leq m_0$, is determined.

Third, one must determine how many and which points on the OC and ASN curves will be calculated. Suppose L values are chosen, denoted by λ_ℓ^*, $\ell = 1,\ldots, L$, such that $\lambda_0 = \lambda_1^* < \lambda_2^* < \ldots < \lambda_L^* = \lambda_1$.

For a given λ_ℓ^*, the first stage density $f_{n_1}(W_{n_1}, Q_{n_1}, R_{n_1})$ may be calculated as follows:

$$f_{n_1}(W_{n_1},Q_{n_1},R_{n_1}) = \left(\frac{1}{n_1^2}\right) \chi^2_{2(n_1-1)}\left[R_{n_1} - \frac{W_{n_1}^2}{n_1} - \frac{Q_{n_1}^2}{n_1}\right]$$

$$\cdot \phi\left(\sqrt{n_1}\left(\frac{W_{n_1}}{n_1}\right)\right) \cdot \phi\left(\sqrt{n_1}\left(\frac{Q_{n_1}}{n_1} - \sqrt{\lambda_\ell^{*'}}\right)\right) \qquad (2.5.1)$$

Note that this density is completely specified by χ^*_ℓ and n_1.

The probabilities of acceptance, rejection, and continuation at stage n_1 (the first stage at which a decision can be made); $P_A{}^{n_1}$, $P_R{}^{n_1}$, and $P_C{}^{n_1}$, may be calculated using the noncentral F distribution (given in equation (1.1.1)) and is shown in appendix A.

To calculate the joint density at the next stage, $f_{n_1+1}(W_{n_1+1},Q_{n_1+1},R_{n_1+1})$, requires utilizing the procedures developed in section 2.3.

As shown in section 2.3, this consists of performing a bivariate integration of the following five dimensional joint density function.

$$f_{n_1}\left(W_{n_1},Q_{n_1},R_{n_1},X_{1n_2},X_{2n_2}\right) = f_{n_1}\left(W_{n_1},Q_{n_1},R_{n_1}\right)$$

$$\cdot \phi\left(X_{1n_2}\right) \cdot \phi\left(X_{2n_2} - \sqrt{\lambda_\ell^{*'}}\right) \qquad (2.5.2)$$

where $n_2 = n_1 + 1$.

This is the joint density of the statistics at stage n; W_{n_1}, Q_{n_1}, R_{n_1}; and the new observations taken at stage $n_2 = n_1 + 1$; X_{1n_2}, X_{2n_2}.

For any given point: $W_{n_1+1} = a$, $Q_{n_1+1} = b$, $R_{n_1+1} = C$; the joint density $f_{n_1+1}(a,b,c)$ is calculated by performing the following bivariate integration.

$$f_{n_1+1}(a,b,c) = \int_{U_L}^{U_U} \int_{Z_L}^{Z_U} f_{n_1}{}^{P}(a-z,b-u,c-z^2-u^2,z,u)\, dz\, du \tag{2.5.3}$$

The limits U_L, U_U, Z_L, and Z_U are dependent upon the particular point (a,b,c) as well as the regions $V_A{}^{n_1}$ and $V_R{}^{n_1}$.

If no decision could be made at stage n_1, these limits are the limits for integrating around the following circle.

$$c - z^2 - u^2 - \frac{(a-z)^2}{n_1} - \frac{(b-u)^2}{n_1} = 0 \tag{2.5.4}$$

and are given in equations (2.3.6) and (2.3.7). Whenever a decision can be made at stage n, the integration region becomes a subset of the points contained inside this circle.

In some cases the integral given in equation (2.5.3) cannot be evaluated as one integral; rather it must be broken up into several pieces, with the overall integral

being the sum of the individual integrals. Equation (2.3.25) is such an example. In such cases, the integration limits for each of the pieces must be determined.

The required integration region for equation (2.5.3) can be one of many. In section (2.3) every possible integration region has been explored; and for each case specific expressions for the U,Z integration limits have been given.

The U,Z integration determination may be best summarized in flowchart format, such as shown in Figure 12

This integration must be determined and performed for all points W_{n_1+1}, Q_{n_1+1}, R_{n_1+1}, thus obtaining the entire density $f_{n_1+1}(W_{n_1+1}, Q_{n_1+1}, R_{n_1+1})$. From this density the probabilities of acceptance $(P_A{}^{n_1+1})$, rejection $(P_R{}^{n_1+1})$, and continuation $(P_C{}^{n_1+1})$ must be calculated. Their calculation requires performing a trivariate integration of the density $f_{n_1+1}(W_{n_1+1}, Q_{n_1+1}, R_{n_1+1})$ over elliptic paraboloids. This is most easily performed as iterated integrals as shown in (2.4.1).

The entire process of obtaining the density, $f_i(W_i, Q_i, R_i)$, from the density, $f_{i-1}^{P}(W_{i-1}, Q_{i-1}, R_{i-1}, X_{1i}, X_{2i})$, and ultimately the probabilities, $P_A{}^i$, $P_R{}^i$, $P_C{}^i$, must be iterated for all stages, $i = n_1 + 2, \ldots m_0$.

The final result, for a given λ_ℓ^*, is the set of probabilities, P_A^i, P_R^i, P_C^i, $i = 2, \ldots m_0$. These probabilities will depend upon the value of λ_ℓ^*. This can easily be seen by noting that both the first step density of equation (2.5.1) as well as the five dimensional density of equation (2.5.2) are both functions of λ_ℓ^*. Therefore, the notation $P_A^i(\lambda_\ell^*)$, $P_R^i(\lambda_\ell^*)$, $P_C^i(\lambda_\ell^*)$, $i = 2, \ldots, m_0$, will be used to denote such a dependence. From these probabilities, the point on the OC and ASN curves for $\lambda = \lambda_\ell^*$ may be calculated. These quantities are calculated as follows:

$$OC(\lambda_\ell^*) = \sum_{L=Z}^{m_0} P_A^i(\lambda_\ell^*) \tag{2.5.5}$$

and

$$ASN(\lambda_\ell^*) = \sum_{L=Z}^{m_0} P_R^i(\lambda_\ell^*) + P_A^i(\lambda_\ell^*) \cdot i = 1 + \sum_{L=Z}^{m_0} P_C^i(\lambda_\ell^*) \tag{2.5.6}$$

Note that, by having all the probabilities $P_A^i(\lambda_\ell^*)$, $P_R^i(\lambda_\ell^*)$, $P_C^i(\lambda_\ell^*)$, other quantities of interest may also be calculated (e.g. variance of DSN, median of DSN, percentile of DSN, etc.).

This entire process has given a single point on the OC and ASN curves. To obtain the next point on the OC and ASN curves the density $f_{n_1}(W_{n_1}, Q_{n_1}, R_{n_1})$ must again be obtained from equation (2.5.1) with $\lambda = \lambda^*_{\ell+1}$. The process of obtaining the density, $f_i(W_i, Q_i, R_i)$, and the probabilities $P_A{}^i(\lambda^*_{\ell+1})$, $P_R{}^i(\lambda^*_{\ell+1})$, $P_C{}^i(\lambda^*_{\ell+1})$, must then be iterated for all stages $i = n_1+1, \ldots, m_0$.

The Direct Method for $K = 2$ SANOVA has been summarized in flowchart format as shown in Figure 13.

FIGURE 12

For any given stage, n: with regions V_A and V_R, the density of the point $(W_n = a, Q_n = b, R_n = c)$ is found by integrating the density of equation (2.3.5) as shown in equation (2.3.8). The integration limits U_U, U_L, Z_U, Z_L may be obtained from the following flowchart.

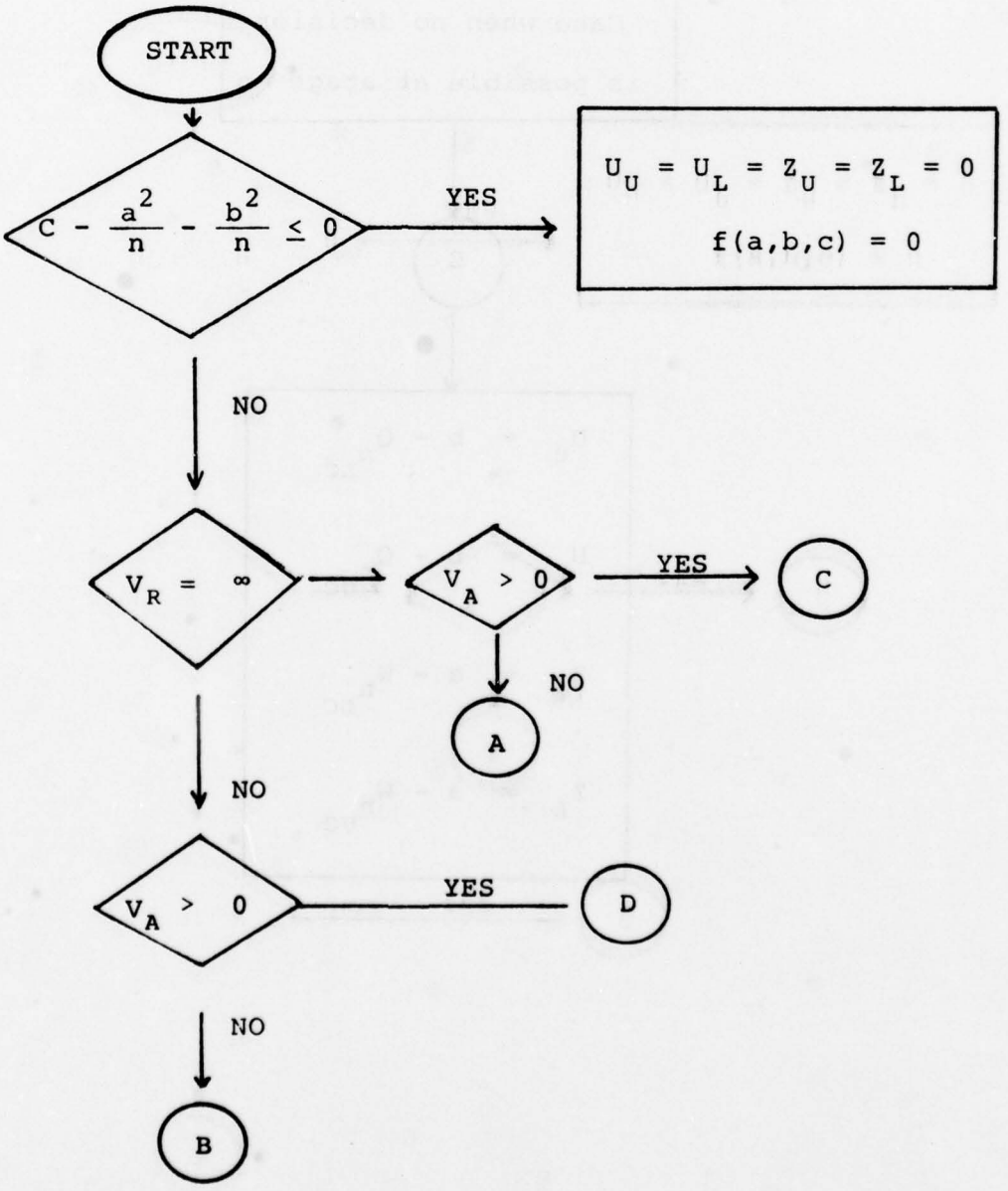


FIGURE 12 (continued)

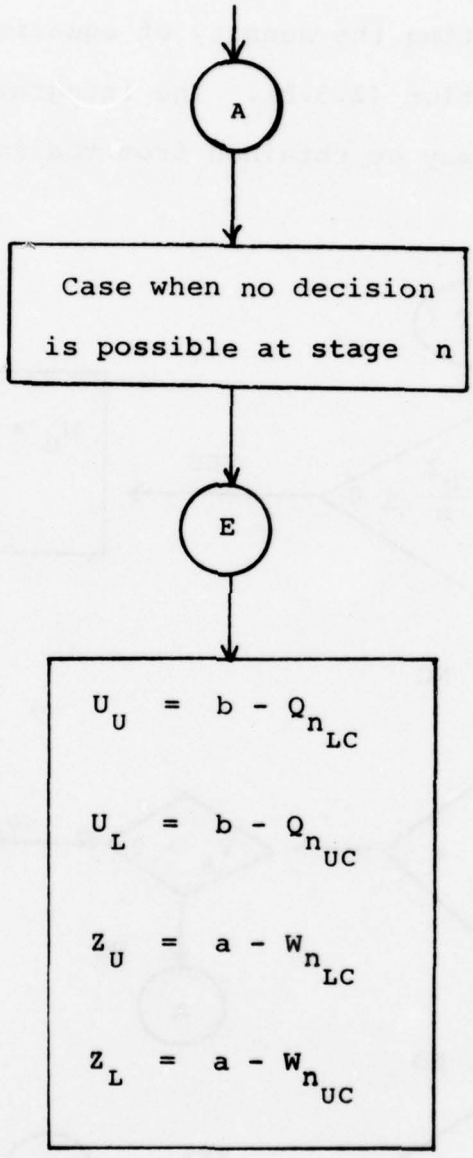


FIGURE 12 (continued)

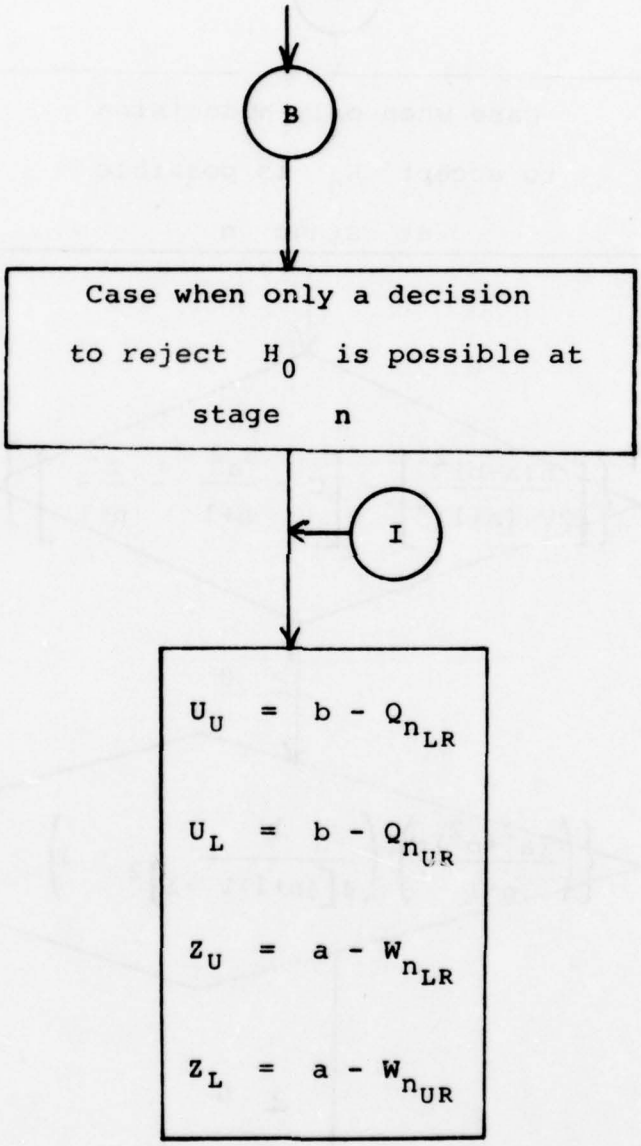


FIGURE 12 (Continued)

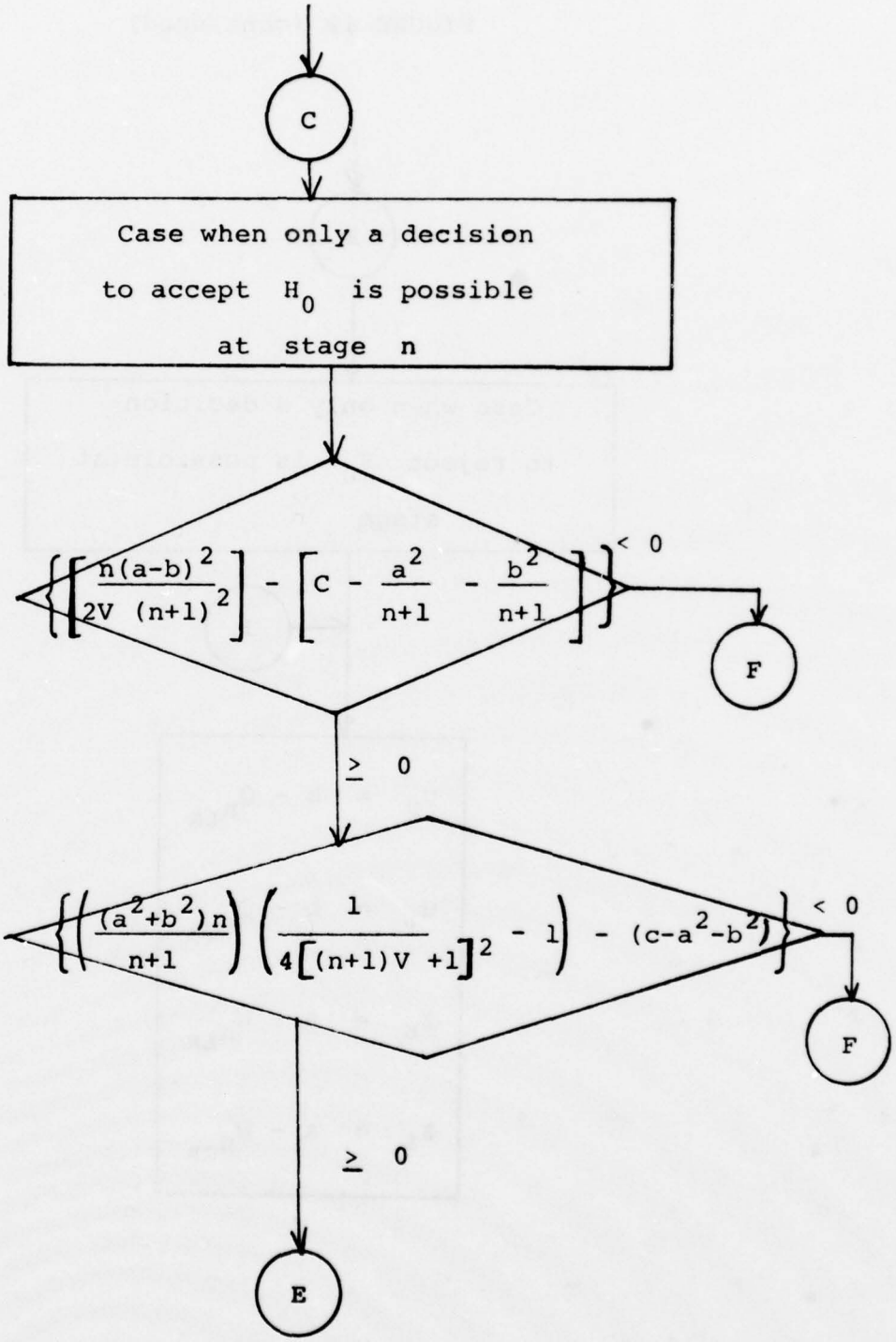
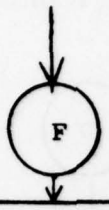


FIGURE 12 (Continued)



The integral must be broken up into at most 4 pieces as given in equation (2.3.25).

The number of pieces will depend upon the sign of $(a+B)^2 - 2(1+\frac{1}{n})(a^2+b^2-c)$

The integration limits for each piece are given by:

Piece	U Limits	Z Limits
1	$U_U = b - Q_{n_{LC}}$	$Z_U = a - W_{n_{LC}}$
	$U_L = b - Q_{n_{LA}}$	$Z_L = a - W_{n_{UC}}$
2	$U_U = b - Q_{n_{UA}}$	$Z_U = a - W_{n_{LC}}$
	$U_L = b - Q_{n_{UC}}$	$Z_L = a - W_{n_{UC}}$
3	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{UA}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{UC}}$
4	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{LC}}$
	$U_L - b - Q_{n_{UA}}$	$Z_L = a - W_{n_{LA}}$

FIGURE 12 (Continued)

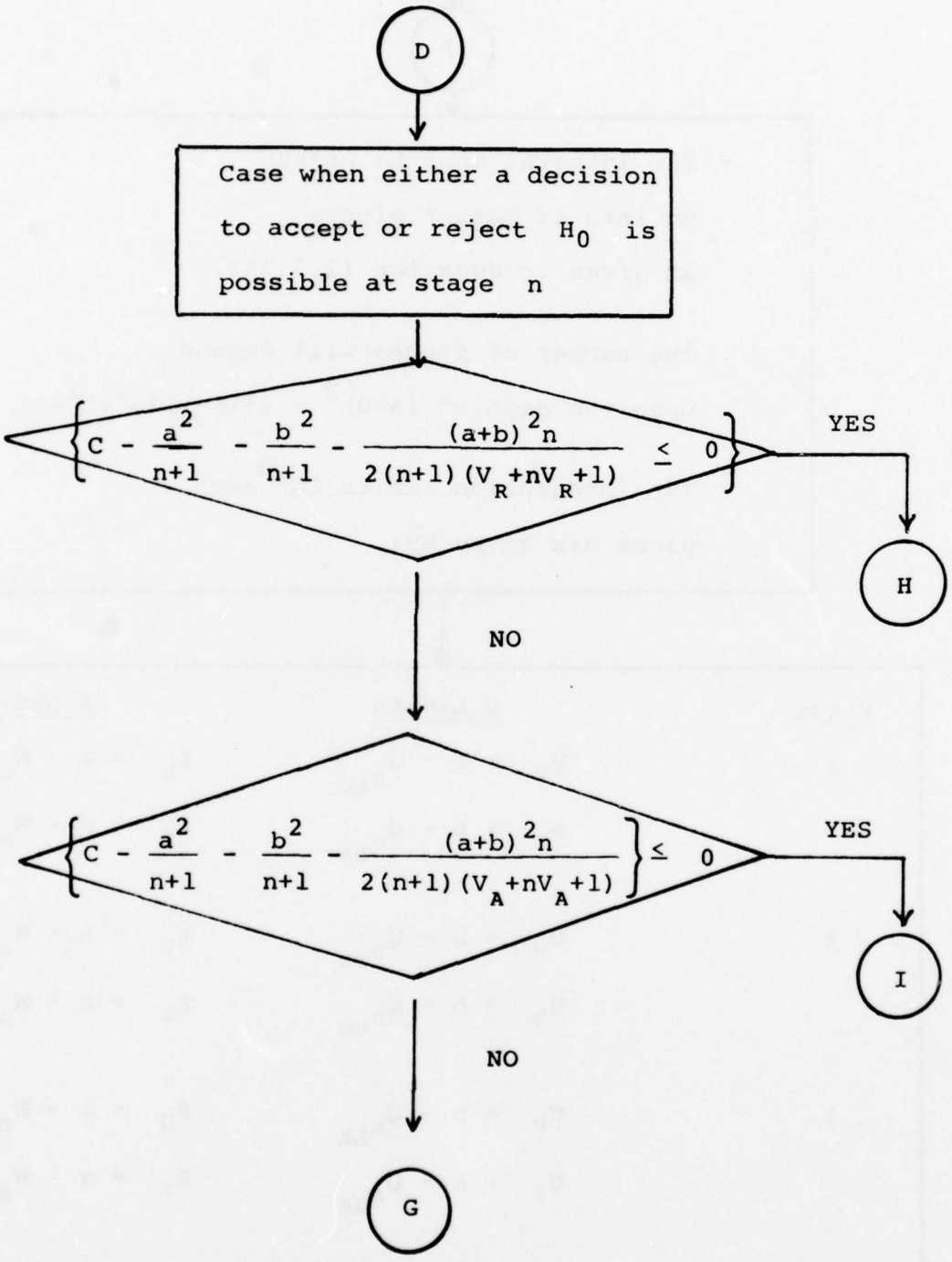


FIGURE 12 (Continued)

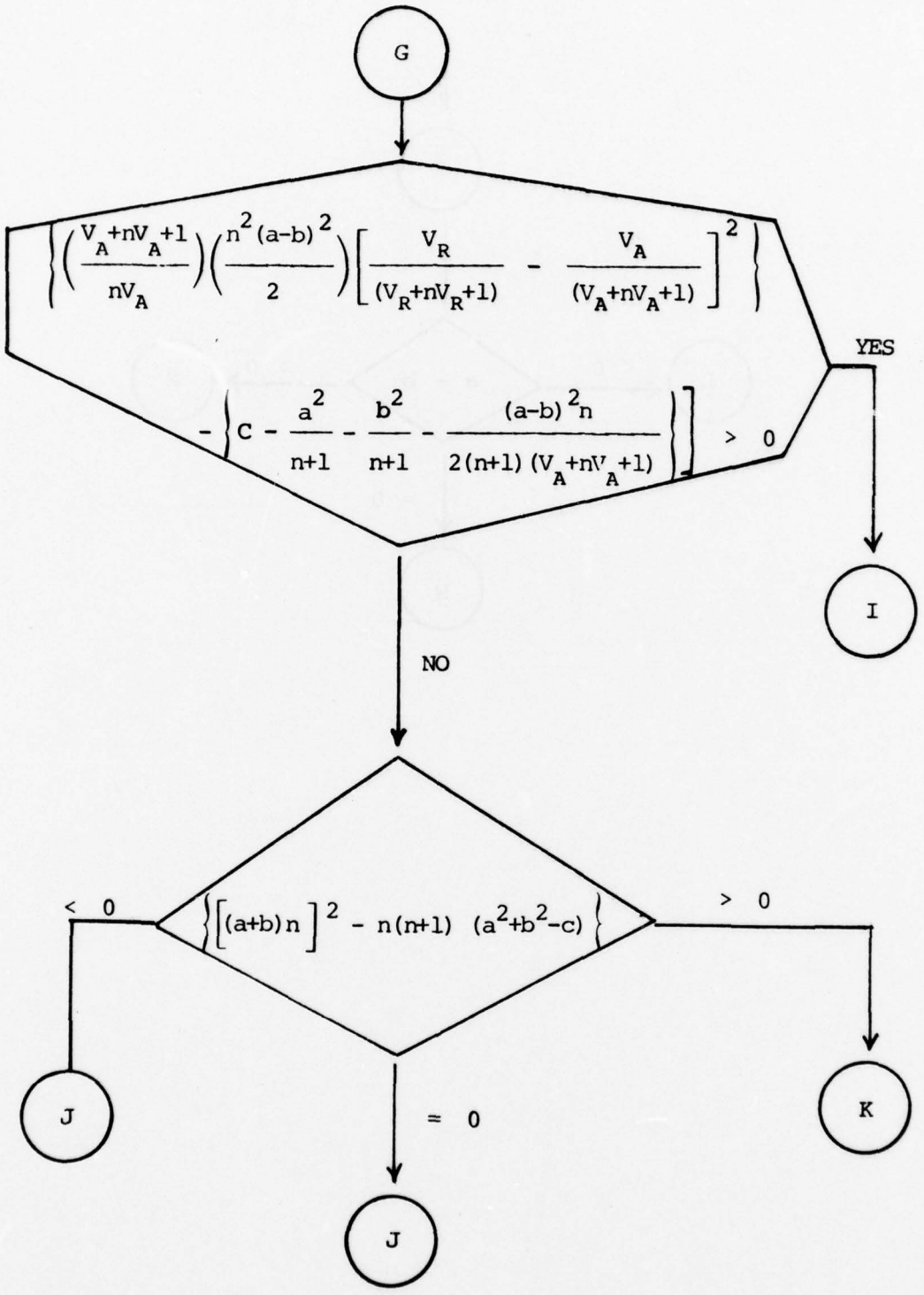


FIGURE 12 (Continued)

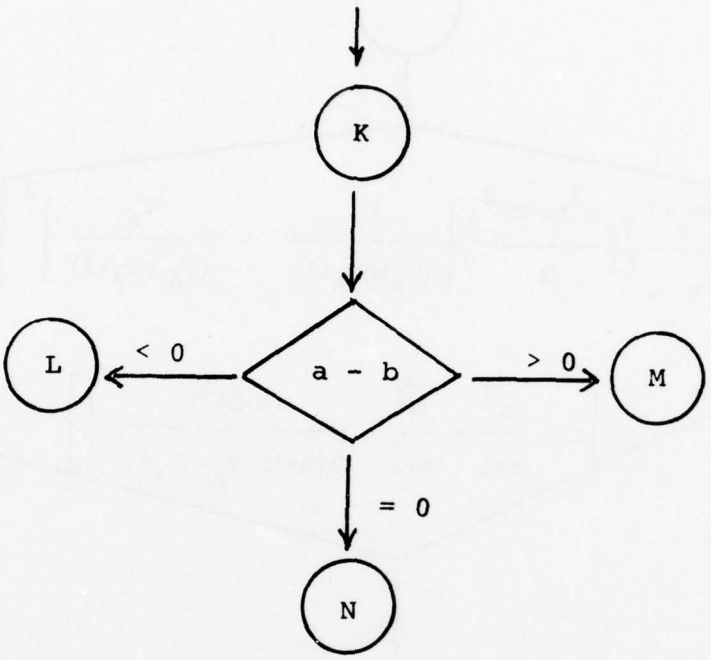


FIGURE 12 (Continued)

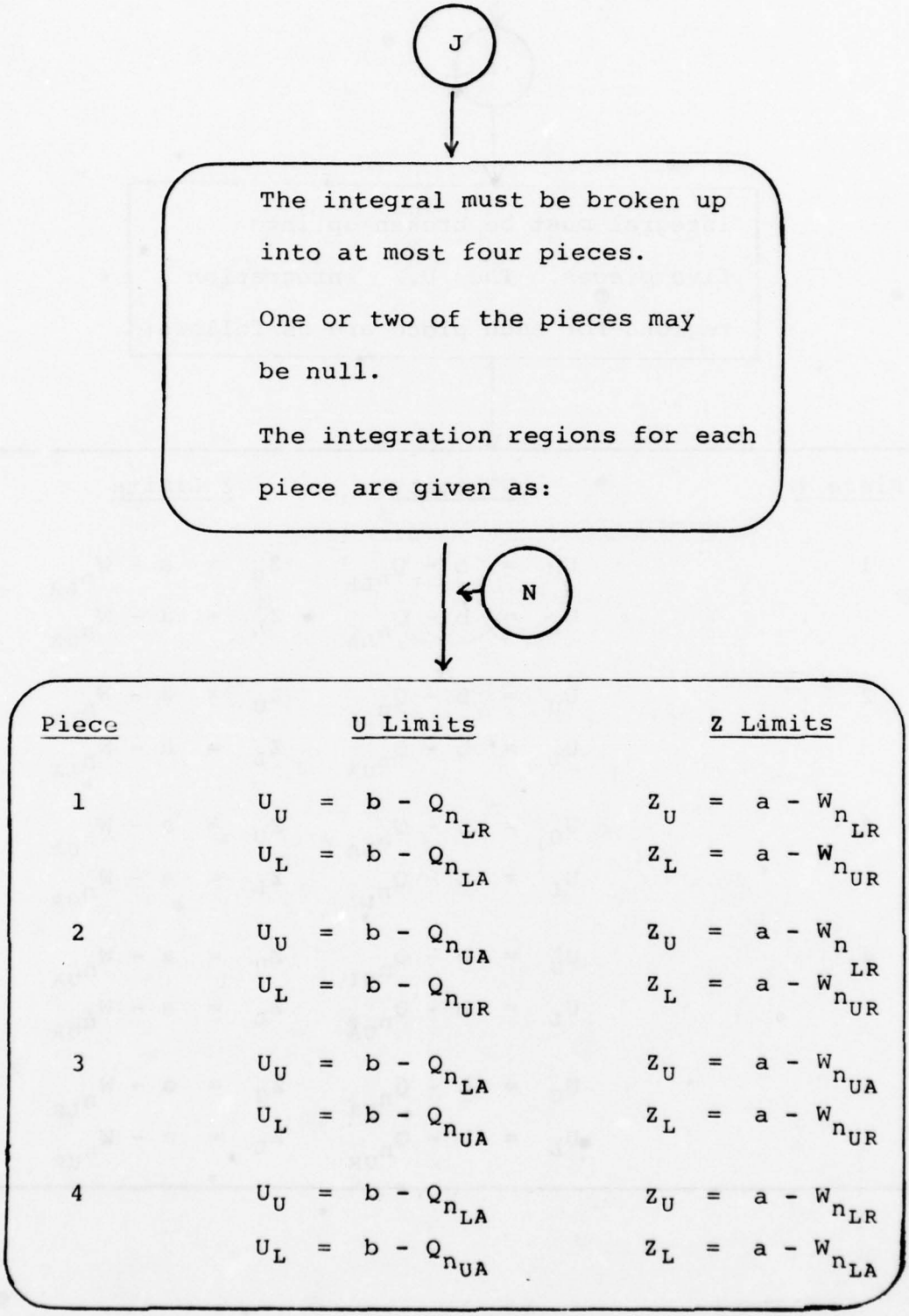
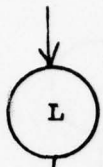


FIGURE 12 (Continued)



Integral must be broken up into five pieces. The U,Z integration regions for each piece are as follows:

Piece #	U Limits	Z Limits
1	$U_U = b - Q_{n_{LR}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{LA}}$	$Z_L = a - W_{n_{UR}}$
2	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{LA}}$
3	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{UA}}$
	$U_L = b - Q_{n_{LI}}$	$Z_L = a - W_{n_{UR}}$
4	$U_U = b - Q_{n_{UI}}$	$Z_U = a - W_{n_{UA}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{UR}}$
5	$U_U = b - Q_{n_{UA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UR}}$	$Z_L = a - W_{n_{UR}}$

FIGURE 12 (Continued)

Integral must be broken up into five pieces. The U,Z integration limits for each piece are given by:

Piece #	U Limits	Z Limits
1	$U_U = b - Q_{n_{LR}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{LA}}$	$Z_L = a - W_{n_{UR}}$
2	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{UA}}$
	$U_L = b - Q_{n_{UA}}$	$Z_L = a - W_{n_{UR}}$
3	$U_U = b - Q_{n_{LA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{LI}}$	$Z_L = a - W_{n_{LA}}$
4	$U_U = b - Q_{n_{UI}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UA}}$	$Z = a - W_{n_{LA}}$
5	$U_U = b - Q_{n_{UA}}$	$Z_U = a - W_{n_{LR}}$
	$U_L = b - Q_{n_{UR}}$	$Z_U = a - W_{n_{UR}}$

FIGURE 13

FLOWCHART

SUMMARY OF DIRECT METHOD

FOR

K=2 SANOVA

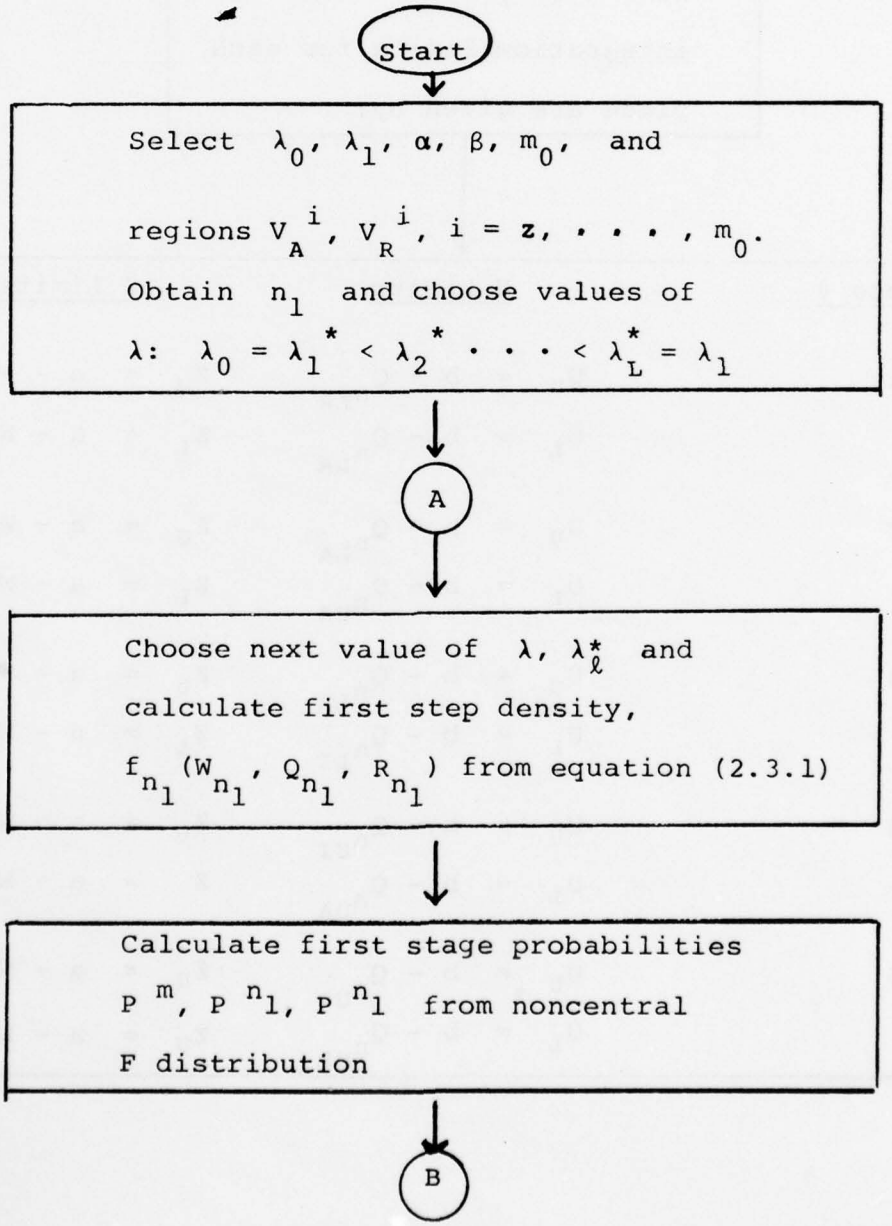


FIGURE 13 (Continued)

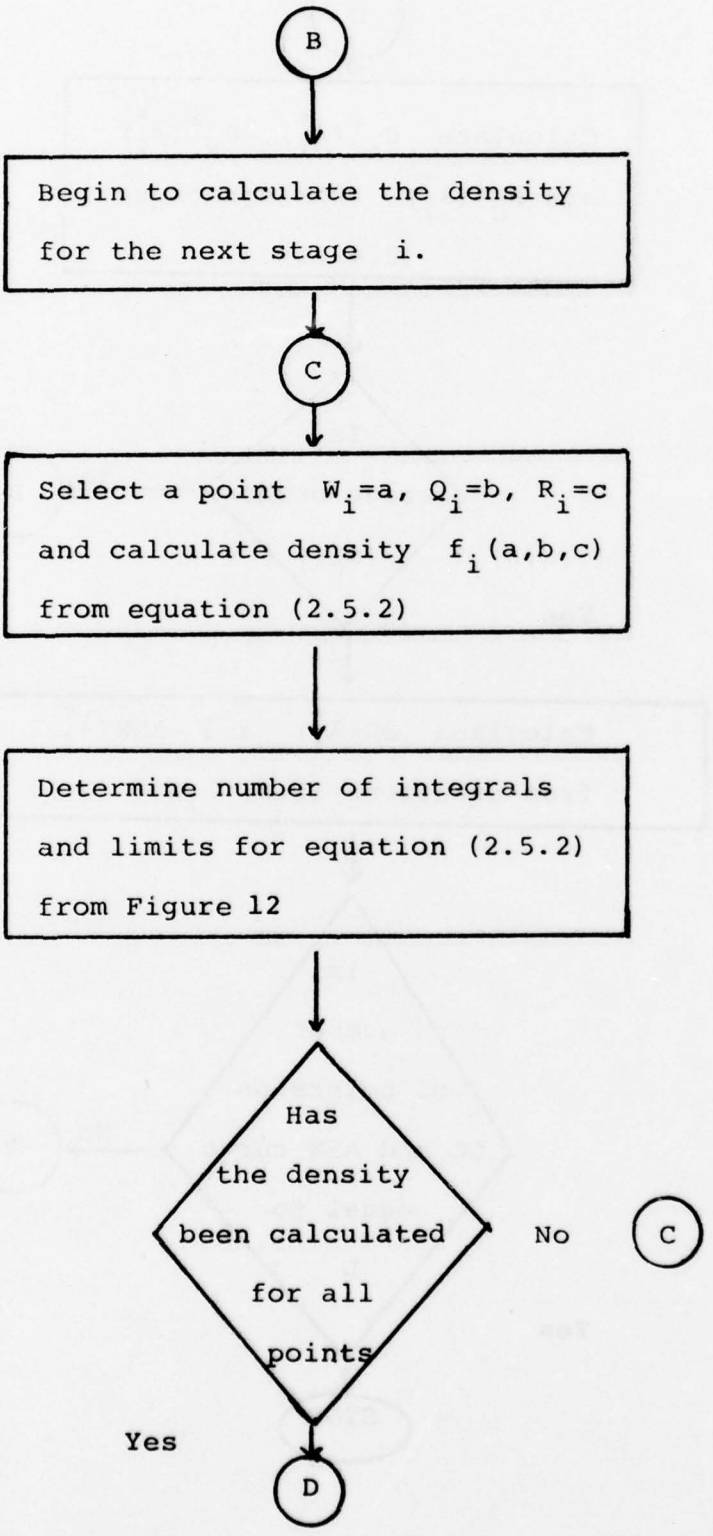
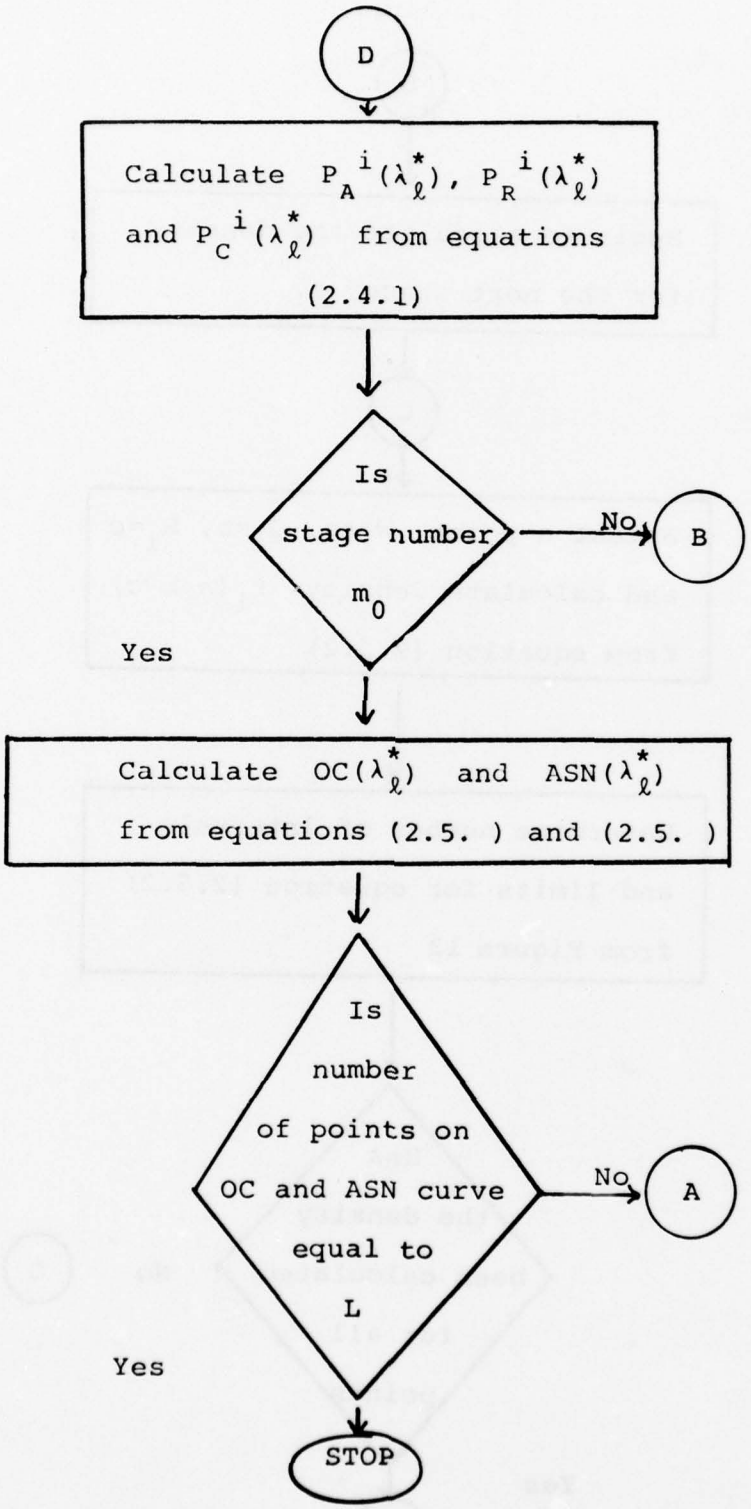


FIGURE 13 (Continued)



2.6 Numerical Methods

The previous sections have given a detailed description and derivation for obtaining the properties of a K=2 SANOVA test by the direct method.

In summary the procedure requires the following steps:

1. For a given value of $\lambda = \lambda^*$, determine the joint density at the first stage at which a decision can be made; $f_{n_1}(W_{n_1}, Q_{n_1}, R_{n_1})$.
2. Calculate the joint density $f_{n_{1+1}}(W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}})$, for all values of $W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}}$. This requires:
   a. Forming the five dimensional joint density $f_{n_1}{}^{P}(W_{n_1}, Q_{n_1}, R_{n_1}, X_{1n_2}, X_{2n_2})$ as given in equation (2.5.2).
   b. Performing the bivariate integration on this five dimensional joint density given in equation (2.5.3).
3. Performing a trivariate integration of the density $f_{n_1+1}(W_{n_1+1}, Q_{n_1+1}, R_{n_1+1})$ to obtain the probabilities of acceptance $(P_A{}^{n_1+1})$, rejection $(P_R{}^{n_1+1})$, and continuation $(P_C{}^{n_1+1})$.

4. Iterating steps 2 and 3 on the density $f_i(W_i, Q_i, R_i)$ for all $i = n_1+2,\ldots,M_0$.

5. Calculating the OC and ASN for $\lambda = \lambda^*$.

6. Repeating steps 1 through 5 for all values of λ^* of interest.

This section will discuss the practical evaluation of the integrals required in steps 2 and 3 of the above procedure.

These integrals will generally be very complicated expressions. For example, wherever no decision can be made, the U, Z region of integration required for step 2 consists of all U, Z contained inside the circle given in equation (2.5.4). The actual U, Z integration limits required are given in equations (2.3.6) and (2.3.7). This amounts to integrating a five dimensional joint density composed of the product of a χ^2 and four normal densities. This integration can be evaluated analytically, yielding the density given at the top of page 2-20. The cases which require integrating aroung ellipses (e.g., equations (2.3.12) - (2.3.14)) or those that require breaking the integral up into several pieces (e.g., equations (2.3.25) - (2.3.28)), generally can not be evaluated analytically.

One approach is to develop a numerical approximation to these bivariate integrals (e.g., series, partial fraction, or continued fraction expansions). Since the integration region required is dependent upon the point a, b, c (for a given set of regions), this approach would yield the following type of piecewise trivariate density for stage n_1+1:

$$f_{n_{1+1}}(W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}}) = \begin{cases} \text{Expression 1} & \text{all } W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}} \in R_1 \\ \vdots & \\ \text{Expression K} & \text{all } W_{n_{1+1}}, Q_{n_{1+1}}, R_{n_{1+1}} \in R_K \end{cases}$$

An analytic expression for the integral required in step 3 with this type of piecewise trivariate density function would probably not exist. Also, if one were to continue along these lines, the density at later stages; $f_i(W_i, Q_i, R_i)$, $i = n_1 + 2,\ldots,m_0$; would become a piecewise function with an infeasible number of pieces.

An alternative method for evaluating these integrals is via numerical integration. The analytic density $f_n(W_n, Q_n, R_n)$ may be represented by a discrete three dimensional grid of points; $f_n(W_{n_i}, Q_{n_j}, R_{n_k})$, $i = 1,\ldots,N_W$, $j = 1,\ldots,N_Q$, $K = 1,\ldots,N_R$; so that for a

given point on this grid; $W_{n_{l+1}} = a = W_{n_{i^*}}$, $Q_{n_{l+1}} = b = Q_{n_{j^*}}$, $R_{n_{l+1}} = c = R_{n_{k^*}}$; the joint density may be approximated by the following expression:

$$f_{n_{l+1}}(a,b,c) \simeq \sum_m \sum_\ell \omega_{1\ell}\, \omega_{2m}\, f^P_{n_{l+1}}(a-Z_\ell,\ b-U_m,\ c-Z_\ell^2-U_m^2,\ Z_\ell,\ U_m)$$

The quantities $\omega_{1\ell}$, ω_{zm} and Z_ℓ, U_m are the required weights and coordinates of the integration scheme employed and depend upon the U, Z region of integration.

Repeating this procedure for all a, b, c contained on this grid yields a new grid representing the density at stage n_{l+1}, $f_{n_{l+1}}(W_{n_i}, Q_{n_j}, R_{n_k})$. From this new grid the probabilities $P_A^{n_1+1}$, $P_R^{n_1+1}$, $P_C^{n_1+1}$ must be obtained. Obtaining these probabilities requires a trivariate integration which can also be done numerically. For example

$$P_A^{n_1+1} \simeq \sum_m \sum_\ell \sum_p \omega_{1m}\, \omega_{2\ell}\, \omega_{3p}\, f_{n_{l+1}}(W_m, Q_\ell, R_p)\ . \tag{2.6.2}$$

This new grid can again be manipulated to obtain the density at stage n_1+2 and ultimately the probabilities $P_A^{n_1+2}$, $P_R^{n_1+2}$ and $P_C^{n_1+2}$. Repeating the procedure to obtain a new grid $f(W_{n_i}, Q_{n_j}, R_{n_k})$ and P_A^n, P_R^n, P_C^n for all $n = n_1+2,\ldots,m_0$, allows the calculation of a point on the ASN and OC curves.

In general, the density at any point not on the grid; say $f_i\ (W_i^*, Q_i^*, R_i^*)$, $i = n_1+2,\ldots,M_0$; must be found by interpolation. Note that this could require the formidable task of interpolating in three dimensions. Thus, it would be desirable to use a grid scheme and integration rule that required a minimum amount of interpolation to evaluate equations (2.4.1) and (2.5.3).

First, consider the following grid scheme:

$$W_{n_i} = \left[W_S + (i-1)\alpha_i\right] h_W \qquad i = 1,\ldots,N_W$$

$$Q_{n_j} = \left[Q_S + (j-1)\beta_j\right] h_Q \qquad j = 1,\ldots,N_Q$$

$$R_{n_k} = \left[R_S + (k-1)\gamma_k\right] h_R \qquad k = 1,\ldots,N_R \qquad (2.6.3).$$

The quantities α_i, β_j, γ_k are all integers chosen such that:

$$W_{n_1} < W_{n_2} < \ldots\ldots < W_{nn_W}$$

$$Q_{n_1} < Q_{n_2} < \ldots\ldots < Q_{nn_Q}$$

$$R_{n_1} < R_{n_2} < \ldots\ldots < R_{nn_R}$$

The choice of the quantities W_S, Q_S, R_S, h_W, h_Q, and h_R will be discussed later.

Many integration rules are available (Davis and Rabinowitz (1967)); but to avoid excessive amounts of interpolation a rule should be chosen which allows the majority of the points to be located on the grid.

For the integration given in (2.6.1) this requires that not only $a - Z_\ell$ and $b - U_m$ be located on W_n, Q_n grid points, but also that $C - Z_\ell^2 - U_m^2$ be located on an R_n grid point. This can be guaranteed if the quantities h_W, h_Q, and h_R are chosen such that:

$$h_R = \Lambda_1 h_W^2 + \Lambda_2 h_Q^2$$

or

$$h_W^2 = \Lambda_3 h_R \quad \text{and} \quad h_Q^2 = \Lambda_4 h_R \,, \tag{2.6.4}$$

where

Λ_1, Λ_2, Λ_3, and Λ_4 are integers.

Using this type of grid and the trapezoid integration rule, equation (2.6.1) becomes:

$$f_i(a,b,c) \simeq \sum_{m=0}^{n_Q} \sum_{\ell=0}^{n_W} \omega_{1\ell}\, \omega_{2m}\, f_{i-1}^P (a-Z_\ell,\ b-U_m,\ c-Z_\ell^2-U_m^2,\ Z_\ell,\ U_m) \tag{2.6.5}$$

where the coordinates Z_ℓ and U_m are given by the following scheme:

$$Z_0 = Z_L \qquad U_0 = U_L$$

$$Z_S = \left[Z_L / h_W\right] \qquad U_S = \left[U_L / h_Q\right]$$

$$Z_F = \left[Z_U / h_W\right] \qquad U_F = \left[U_U / h_Q\right]$$

$$N_W - 1 = Z_F - Z_S + 1 \qquad N_Q - 1 = U_F - U_S + 1$$

$$Z_\ell - 1 = Z_S + (\ell - 1)h$$
$$\ell = 1, \ldots, N_W - 2$$

$$Z_{n_{W-1}} = Z_F$$

$$Z_{n_W} = Z_U$$

$$U_m = U_S + (m-1)h_W$$
$$m = 1, \ldots, N - 2 \qquad (2.6.6)$$

$$U_{n_{Q-1}} = U_F$$

$$U_{n_Q} = U_U$$

with

$$[X] \equiv \text{sign } (X) \cdot \{\text{greatest integer in } 1 \times 1\}.$$

The weights $\omega_{1\ell}$ and ω_{2m} are given by:

$$\omega_{10} = \tfrac{1}{2}|(Z_L - Z_0)|$$

$$\omega_{11} = \tfrac{1}{2}|(Z_S - Z_0)| + \tfrac{1}{2}h_W$$

$$\omega_{1\ell} = \tfrac{1}{2}h_W \qquad \ell = 2, \ldots, N_W - 2$$

$$\omega_{1\ell} = \tfrac{1}{2}h_W + \tfrac{1}{2}|(Z_U - Z_F)| \qquad \ell = N_W - 1$$

$$\omega_{1\ell} = \tfrac{1}{2}|(Z_U - Z_F)| \qquad \ell = N_W$$

and

$$\omega_{20} = \tfrac{1}{2}|(U_L - U_S)|$$

$$\omega_{21} = \tfrac{1}{2}|(U_S - U_0)| + \tfrac{1}{2}h_W$$

$$\omega_{2m} = \tfrac{1}{2}h_Q \qquad m = 2,\ldots,N_Q-2$$

$$\omega_{2m} = \tfrac{1}{2}h_Q + \tfrac{1}{2}|(U_U - U_F)| \qquad m = N_W-1$$

$$\omega_{2m} = \tfrac{1}{2}|(U_U - U_F)| \qquad m = N_W$$

With this grid structure and integration scheme the density of some points may still need to be obtained by interpolation. Any values of Z^*, U^* that result in the point $(a-Z^*, b-U^*, c-Z^{*2}-U^{*2})$ not to be on a (W,Q,R) grid point will require that the five dimensional density $f_i^P(\quad)$ be obtained by interpolation. For example, there is no guarantee that the endpoints Z_L, Z_U, U_L and U_U will lie on a grid point. However, in such cases, the task of interpolation may be simplified by considering the form of the five dimensional density $f_i^P(\quad)$.

As shown in equation (2.5.2) the five dimensional joint density is given by:

$$f_i^P(a-Z,\ b-U,\ c-Z^2-U^2,\ Z,\ U)$$

$$= f_{i-1}(a-Z,\ b-U,\ c-Z^2-U^2) \cdot \phi(Z) \cdot \phi(U-\sqrt{\lambda_\ell^*}).$$

Whenever interpolation is required to evaluate $f_i^P(\quad)$, it need only be performed in two or three dimensions on the

density $f_{i-1}(\quad)$, since both ϕ's can be calculated exactly for any value of U and Z.

In other words, when interpolation is required

$$f_i^P(a-Z,\ b-U,\ c-Z^2-U^2,\ Z,\ U) \approx E^* \phi(Z) \cdot \phi\left(U-\sqrt{\lambda_\ell^*}\right)$$

where E^* is the interpolated value of the density $f_{i-1}(a-Z,\ b-U,\ c-Z^2-U^2)$.

For a given point a^*, b^*, c^* not on a (W,Q,R) grid point at stage i-1, the density $f_{i-1}(a^*, b^*, d^*)$ may be approximated by trivariate linear interpolation. This involves the following approximation:

$$f_{i-1}(a^*,b^*,c^*) = P^* \approx \sum_{\ell=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} f_{i-1}(a_\ell,\ b\ ,\ c\ )\alpha_\ell \beta_j \gamma_k \qquad (2.6.8)$$

where

$$a_1 = a^*/h_W + 1 \ (\text{sign } (a^*)$$

and

$$a_2 = a^*/h_W$$

and

$$\alpha_1 = \frac{a^*-a_2}{a_1-a_2} \quad , \quad \alpha_2 = \frac{a^*-a_1}{a_2-a_1}$$

and analagous expressions for the other quantities.

This should give fairly good approximations for small values of h_W, h_Q, and h_R. For large values of these quantities the result could be meaningless (i.e., $f_i(a^*, b^*, c^*) < 0$ or $f_i(a^*, b^*, c^*) > 1$), and should be modified in such cases. The modifications are of the following form:

$$f_i(a^*, b^*, c^*) = \begin{cases} P^* & \text{if} \quad 0 \leq P^* \leq 1 \\ 0 & \text{if} \quad P^* < 0 \\ \max(f_i(a_\ell, b_j, c_k) & \text{if} \quad P^* \geq 1 \end{cases}$$

By using the trapezoid rule and trivariate interpolation, the density $f_i(a, b, c)$ may be calculated. This must be repeated for all a, b, c contained on the grid. This will result in a new grid representing the density at stage i. From this new grid, the probabilities P_h^i, P_R^i, and P_C^i must be calculated. These probabilities can also be calculated with a trapezoid rule integration scheme as given in (2.6.2).

In practice the following quantities must be specified:

(1) The grid sizes h_W, h_Q, h_R .

(2) The end points of the grid: W_S, W_F, Q_S, Q_F, R_S, R_F .

As in most numerical problems, the best choice of the grid sizes will depend upon the particular problem (i.e., V_A^i, V_R^i, and m_0). One approach to this problem is to start the procedure with a coarse grid and obtain answers; the procedure may then be redone using a finer grid and new answers obtained. This process is iterated until the results converge to answers accurate to the desired number of digits. One should note that the number of calculations required for each additional iteration increases exponentially. For example, suppose a grid is constructed, using grid sizes h_W, h_Q, and $h_R = h_W^2+h_Q^2$ and α_i, β_j, γ_k of (2.6.3) all equal to unity. Halving the h_W and h_Q grid sizes will result in an eight-fold increase in the total number of points on the grid. The density for each of these points must be calculated for each stage, which requires a bivariate integration for each point at each stage.

The grid end points must be chosen so as to exclude only a minute fraction of the density for all stages: $i = n_1,\ldots,m_0$. This amounts to choosing the quantities

W_S, W_F, Q_S, Q_F and R_S, R_F such that all points, W_n, Q_n. R_n, on the grid lie within these ranges, i.e.,

$$W_S \leq W_b \leq W_F$$

$$Q_S \leq Q_n \leq Q_F$$

$$R_S \leq R_n \leq R_F .$$

In most cases, the size of the required grid (i.e., W_S, W_F, Q_S, Q_F, R_S and R_F) is directly proportional to the value of m_0.

Since W_{n_1} (n_1 being the first stage at which a decision can be made) is distributed normally with mean zero and standard deviation $\sqrt{n_1}$, a W_n range of the following type:

$$W_S = -6\left[\sqrt{n_1}\ /h_W\right]*h_W$$

$$W_F = 6\left[\sqrt{n_1}\ /h_W\right]*h_W \text{ , where } [\ ] \equiv \text{greatest integer}$$

should be sufficient for the grid at stage n_1. However, if the regions were such that no decision could be made until stage m_0, $W_{m_0} \sim N(0, \sqrt{m_0})$. Thus in order to insure that the grid is large enough, the following range should be used:

$$W_S = -6\left[\sqrt{m_0}\ /h_W\right]*h_W$$

$$W_F = +6\left[\sqrt{m_0}\ /h_W\right]*h_W$$

(2.6.9).

Employing similar logic to the Q dimension yields the following range:

$$Q_S = \min \left\{ \left[(\sqrt{n_1 \lambda} - 6\sqrt{n_1}\,)/h_Q \right] * h_Q,\ \left[(\sqrt{m_0 \lambda} - 6\sqrt{m_0}\,)/h_Q \right] * h_Q \right\}$$

$$Q_F = \max \left\{ \left[(\sqrt{n_1 \lambda} + 6\sqrt{n_1}\,)/h_Q \right] * h_Q,\ \left[(\sqrt{m_0 \lambda} + 6\sqrt{m_0}\,)/h_Q \right] * h_Q \right\}$$

where $[\ ]$ $\equiv$ greatest integer. (2.6.10).

Since R must always be greater than the quantity $1/n(W^2 + Q^2)$, the R points for which

$$R_{m_0} < \frac{1}{m_0} \left(W_0^2 + Q_{m_0}^2 \right),$$

need not be contained on the grid. Thus the range of R will depend upon the values of W and Q, and the overall grid structure becomes that of a cone as shown in Figure 14. An R range sufficient for the density $f_i(W_i, Q_i, R_i)$ for all i, is given by:

$$R_S = \left[\frac{1}{m_0} (W_{n_K}^2 + Q_{n_j}^2)/h_R \right] * h_R$$

$$R_F = R_S + \left[\chi^2_{99.9} \, (2m_0 - 2)/h_R \right] * h_R \qquad (2.6.11).$$

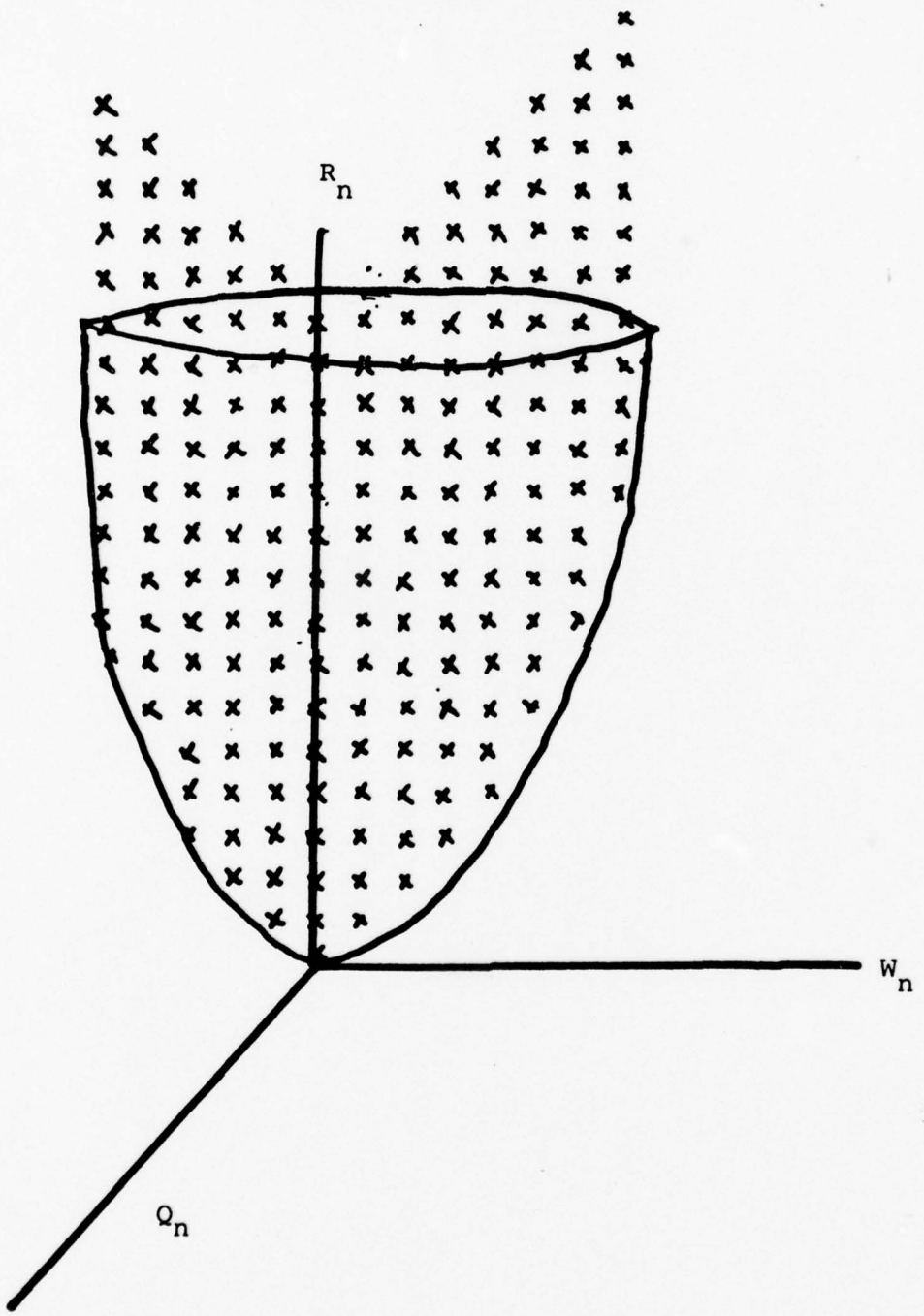
where $[\ ] \equiv$ greatest integer.

A grid of this form and size will allow for sufficient accuracy in the calculation of the OC and ASN curves.

In conclusion, this section has presented a procedure for implementing the theory of the previous sections.

FIGURE 14

STRUCTURE OF NUMERICAL GRID
FOR DIRECT METHOD IMPLEMENTATION



Since the density $f_i(W_i, Q_i, R_i)$ can not be expressed in a closed form for $i > n_1$, this section has discussed a numerical procedure which allows the implementation of the theory. The numerical procedure consists of:

1. Representing the density $f_i(W_i, Q_i, R_i)$ by a discrete 3-dimensional grid of points. The grid is shown in Figure 14 and is described mathematically by equations (2.6.3). The quantities R_S, R_F, Q_S, Q_F, W_S, W_F, h_R, h_W, h_a are given by equations (2.6.4) and (2.6.9) - (2.6.11).
2. "Carrying" this grid from stage to stage. The grid at stage $i-1$ is used to calculate a new grid for stage i, which represents the density $f_i(W_i, Q_i, R_i)$. To calculate the density of any point on this grid at stage i requires performing the bivariate integration of equation (2.5.3). However, the integration is now performed numerically. When the trapezoid integration rule is used, the calculation is given by equations (2.6.5) - (2.6.7).
3. After the density of all points at stage i has been calculated, the grid is then again numerically integrated to obtain P_A^i, P_R^i, P_C^i. This is calculated by the procedure shown in equation (2.6.2).

Since the density at stage i-1 is known only at the points on the grid, the density at points not on the grid must be obtained by interpolation. This can be done by three dimensional linear interpolation as given in equation (2.6.8).

The methods discussed in this section are only feasible if performed on an electronic computer. Appendix C discusses a program developed to calculate several points on the OC and ASN curves for any k=2 SANOVA test.

2.7 CONCLUSION

This chapter of the thesis has derived a procedure for obtaining the OC and ASN curves of a k=2 SANOVA test. The procedure is the first to yield exact results.

Section (2.3) involved the theoretical derivation of the procedure, which has been summarized in Figures 12 and 13 of Section (2.5). Also, Section (2.6) contained a discussion of a numerical approach for implementing the procedure. Appendix C contains a computer program which calculates the OC and ASN curves via the methods discussed in this chapter.

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APPENDIX A

POWER CALCULATIONS FOR A FIXED SAMPLE ANOVA TEST

As shown in Section (1.1) of the thesis, the fixed sample test utilizes the statistic F_{cal}, where

$$F_{cal} = \frac{\sum_{i=1}^{K} n_i \, (\bar{X}_i - \bar{\bar{X}})^2/(K - 1)}{\sum_{i=1}^{K} \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i)^2/(N - K)}$$

with

$$N = \sum_{i=1}^{K} n_i$$

$$\bar{X}_i = n_i^{-1} \sum_{j=1}^{n_i} X_{ij}$$

$$\bar{\bar{X}} = N^{-1} \sum_{i=1}^{K} \sum_{j=1}^{n_i} X_{ij}$$

For a test of K means with $n_i = n$ observations from each population

$$F_{cal} \sim F_{K-1,K(n-1)}(n\lambda)$$

where

$$\lambda = \frac{\sum_{i=1}^{K} (\mu_i - \bar{\mu})^2}{\sigma^2}$$

$$\mu = \sum_{i=1}^{K} \mu_i / K$$

and $F_{K-1,K(n-1)}(n\lambda)$ is a noncentral F variate as defined in Section (1.1).

The ANOVA test is usually a test of the following hypotheses:

$$H_0: \lambda = 0 \quad \text{vs.} \quad H_1: \lambda \geq \lambda'$$

The decision criterion of the test is as follows:

(1) Accept H_o if $F_{CAL} < F^*_{K-1,K(n-1),\alpha}) = \alpha$.

The quantity α corresponds to the acceptable probability of a Type-I error.

The choice of any two of the three quantities (β (magnitude of the Type-II error), n, λ') completely determines the third.

The OC curve of the test is in terms of the parameter λ, and is defined as:

$$OC(\lambda^*) = \Pr(\text{accepting } H_0 \mid \lambda = \lambda^*)$$

$$= \Pr(F_{CAL} < F^*_{K-1,K(n-1),\alpha} \mid \lambda = \lambda^*)$$

$$= \Pr(F_{CAL} < F^*_{K-1,K(n-1),\alpha} \mid F_{CAL\ K-1,K(n-1)}(n\lambda^*))$$

$$= \int_0^{F^*_{K-1,K(n-1),\alpha}} f(F_{K-1,K(n-1)}(n\lambda^*)\ dF_{K-1,K(n-1)}(n\lambda^*)$$

where $f(F_{K-1,K(n-1)}(n\lambda^*))$ is the density of a noncentral F variate with $K-1, K(n-1)$ degrees of freedom and noncentral parameter $n\lambda^*$.

In order to calculate this integral the noncentral F distribution must be integrated. This integration can be expressed in terms of an infinite series of multiples of incomplete beta function ratios in the following manner:

$$OC(\lambda^*) = \sum_{j=0}^{\infty} \left(\frac{[\tfrac{1}{2}n\lambda^*]^j}{j!}\ e^{-\frac{1}{2}n\lambda^*} \right) I_g(\tfrac{1}{2}(K-1)+j,\ \tfrac{1}{2}K(n-1))$$

where $$g = \frac{(K-1)F^*_{K-1,K(n-1),\alpha}}{\left[K(n-1) + (K-1)\ F^*_{K-1,K(n-1),\alpha}\right]} \qquad (A.1)$$

and

$$I_x(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^x t^{a-1}(1-t)^{b-1}\, dt$$

the incomplete beta function.

Thus $OC(\lambda^*)$ may be calculated by summing terms in the series until the magnitude of a term is less than or equal to some ε.

The incomplete beta function cannot be evaluated analytically, so must be done numerically. One method is that of continued fractions. The incomplete beta function's continued fraction expansion was obtained by Aroian (1941) and is given in Abramowitz and Stegum (1969).

An approximation to the cumulative distribution function of the noncentral F distribution was given by Tiku (1966). His approximation consists of fitting the distribution of $F_{\nu_1,\nu_2}(\lambda)$ by that of $(b + cF_{\nu_1',\nu_2})$; choosing b, c, and ν_1' so as to make the first three moments agree. The values which do this are:

$$\nu_1' = \tfrac{1}{2}(\nu_2-2)\left[\sqrt{\frac{H^2}{H^2-4K^3}} - 1\right] \qquad (A.2)$$

$$c = (\nu_1'/\nu_1)(2\nu_1+\nu_2-2)^{-1}(H/K)$$

$$b = -\nu_2(\nu_2-2)^{-1}(c-1-\lambda\ \nu_1^{-1})$$

where

$$H = 2(\nu_1+\lambda)^3 + 3(\nu_1+\lambda)(\nu_1+2\lambda)(\nu_2-2) + (\nu_1+3\lambda)(\nu_2-2)^3$$

and

$$K = (\nu_1+\lambda)^2 + (\nu_2-2)(\nu_1+2\lambda)$$

so that

$$\Pr(F_{\nu_1,\nu_2}(\lambda) \leq f_0) \approx \Pr(b+cF_{\nu_1,\nu_2} \leq f_0)$$

$$= \Pr\left(F_{\nu_1,\nu_2} \leq \frac{f_0-b}{c}\right) \qquad \text{(A.3)}$$

This approximation simply requires a method for evaluating the cumulative distribution function of a central F with ν_1' and ν_2 degrees of freedom, which from above can be calculated as:

$$\Pr(F_{\nu_1', \nu_2} \leq X) = I_{\nu_1 X/(\nu_2+\nu_1' X)}(\tfrac{1}{2}\nu_1, \tfrac{1}{2}\nu_2') \ . \tag{A.4}$$

The computer program contained at the end of Appendix B uses this approximation to calculate the OC curve of a fixed sample test with specified values of α, β and λ'.

APPENDIX B

OBTAINING WALD REGIONS FOR A SANOVA TEST

As discussed in the thesis, a SANOVA test is conducted using the test statistic F_n of equation (1.2.1) or the simpler statistic V_n, where

$$V_n = \frac{(K-1)}{(N-K)} F_n .$$

At each stage this statistic is calculated and compared with the quantities V_A^n and V_R^n; such that at any stage i:

(1) H_0 is accepted if $V_i \leq V_A^i$

(2) H_1 is accepted if $V_i \geq V_R^i$.

The regions V_A^i, V_R^i, are usually chosen so that the Type-I and Type-II errors are approximately equal to the risks acceptable to the experimenter (α and β). The regions developed by Wald are the most commonly used.

For a given set of quantities α, β, K, λ_0, and λ_1, Wald regions V_A^n and V_R^n are obtained as the solutions of the following equations:

$$\frac{\exp\left\{-\frac{n}{2}(\lambda_1-\lambda_0)\right\} M\left[\frac{Kn-1}{2}, \frac{K-1}{2}, \frac{m\lambda_1 V_A^n}{2(1+V_A^n)}\right]}{M\left[\frac{Kn-1}{2}, \frac{K-1}{2}, \frac{m\lambda_0 V_A^n}{2(1+V_A^n)}\right]} = \frac{\beta}{1-\alpha}$$

and

$$\frac{\exp\left\{-\frac{n}{2}(\lambda_1-\lambda_0)\right\} M\left[\frac{Kn-1}{2}, \frac{K-1}{2}, \frac{m\lambda_1 V_R^n}{2(1+V_R^n)}\right]}{M\left[\frac{Kn-1}{2}, \frac{K-1}{2}, \frac{m\lambda_0 V_A^n}{2(1+V_R^n)}\right]} = \frac{1-\beta}{\alpha}$$

where $M(X, Y, Z)$ is the confluent hypergeometric function given in Section (1.1) and discussed by Stater (1960).

These quantities are obtained by solving the above equations by a Newton-Raphson root solving technique (Carnahan, et al (1969)).

In some cases, i.e., for small values of n, a root does not exist for the equations above. In such cases it is not possible to make a decision at that stage.

The following pages contain a listing of a computer program which will calculate regions for any given values of α, β, K, λ_0 and λ_1.

Tables of such regions have been worked out by Ghosh and West (1967) for selected values of α, β, λ_0, and λ These regions however are only given for every fifth stage. Thus the following computer program also allows the Ghosh regions to be read in, and the missing region values calculated via Lagrangian interpolation (Ghosh and West (1967)).

```
        B 5 7 0 0    F O R T R A N    C O M P I L A T I O N   XV.3.00.   THURS

FILE  5=CARD.UNIT=READER
FILE  6=PRINTER.UNIT=PRINTER
C  ***************************************************************************
C                    THIS PROGRAM ALLOWS
C                    DETERMINATION OF FIXED SIZE ANOVA TEST
C                    THIS PROGRAM WILL FIND A CRITICAL VALUE
C                    C≥0 AND THE SMALLEST INTEGER N FOR GIVEN
C                    VALUES OF ALPHA AND BETA
C                    IF V<C THEN HO IS ACCEPTED AND IF V>C HO IS REJECTED
C  ***************************************************************************
C
C
                                                                 START OF SEGMEN
      DIMENSION   BOUND(2).XLIN(2).REG(50.2)
      DOUBLE PRECISION VALO.HYP1.HYP2.HYP3.HYP4.VAL2.VALST.FX.XLN.AUX1
     1  .AUX2 .XNXT.FXNXT.XEVAL   .FPX
      REAL LAMO.LAM1
      COMMON EPS
      TAU(Z)=((Z-0.5)*ALOG(Z))-Z+(0.5*ALOG(6.283185))+(1.0/(12.0*Z))
     1-(1.0/(360.0*(Z**3.0)))+(1.0/(1260.0*(Z**5.0)))-(1./(1680.*(Z**7.0
     2)))
      H1(GRN.SS.CP)=2.0*(((GRN-1.0)+(SS*CP))**3.0)
      H2(GRN.SS.CP)=3.0*((GRN-1.0)+(SS*CP))*((GRN-1.0)+(2.0*SS*CP))*(((G
     1RN*(SS-1.0))-2.0))
      H3(GRN.SS.CP)=((GRN-1.0)+(3.0*SS*CP))*(((GRN*(SS-1.0))-2.0)**2.0)
      CON2(GRN.SS.CP)=(((GRN-1.0)+(SS*CP))**2.0)+(((GRN*(SS-1.0))-2.0)
     2*((GRN-1.0)+(2.0*SS*CP)))
      CON3(GRN.SS.CP.H)=((GRN*(SS-1.0))/((GRN*(SS-1.0))-2.0))*(H-(((GRN-
     31.0)+(SS*CP))/(GRN-1.0)))
      EPS=1.0 E-8
      IREAD=5
      IRITE=6
      READ(IREAD.1) ALPHA.BETA.LAMO.LAM1.DEGF
1     FORMAT(5F10.4)
      WRITE(IRITE.701)
701   FORMAT(1H1.20X."FIXED SAMPLE ANOVA TEST")
      WRITE(IRITE.702)
702   FORMAT(/.20X."**************************")
      WRITE(IRITE.703)  DEGF
703   FORMAT(///.24X."K=".F3.1.2X."GROUPS")
      READ(IREAD.713) SAM
13    FORMAT(F8.2)
      IF(LAM1=0.0  .AND. SAM  .GT. 0.0)  GO TO 21
      WRITE(IRITE.704)  LAMO.LAM1
704   FORMAT(////.13X.9HHO:LAMO =.F6.2.2X."VS".2X.9HH1:LAM1 =.F6.2)
      WRITE(IRITE.705)  ALPHA.BETA
705   FORMAT(//.20X."ALPHA =".F5.2.6X."BETA =".F5.2)
      DO 10 N=3.1000
      SAM=FLOAT(N)
      IF(LAMO=0.0)  GO TO 5
      H=H1(DEGF.SAM.LAMO)+H2(DEGF.SAM.LAMO)+H3(DEGF.SAM.LAMO)
      CONK=CON2(DEGF.SAM.LAMO)
      E=(H**2.0)/(CONK**3.0)
      B=((DEGF*(SAM-1.0))-2.0)*(SQRT(E/(E-4.0))-1.0)
      B=B*0.5
      V=(B/(DEGF-1.0))*(H/CONK)*(1.0/((2.0*B)+(DEGF*(SAM-1.0))-2.0)  )
      C=CON3(DEGF.SAM.LAMO.V)
      T1=(DEGF*(SAM-1.0))/2.0
      T2=B/2.0
      GO TO 7
5     T1=0.5*DEGF*(SAM-1.0)
```

```
      T2=0.5*(DEGF-1.0)
      B=1.0
7     YO=BETINC(1,T1,T2,ALPHA)
      FO=((V*DEGF*(SAM-1.0)*(1.0-(YO*B)))/(YO*B))-C
      IF(LAMO=0.0) FO=(DEGF*(SAM-1.0)*(1.0-YO))/((DEGF-1.0)*YO)
      T1=(DEGF*(SAM-1.0))/2.0
      HP=H1(DEGF,SAM,LAM1)+H2(DEGF,SAM,LAM1)+H3(DEGF,SAM,LAM1)
      CONKP=CON2(DEGF,SAM,LAM1)
      EP=(HP**2.0)/(CONKP**3.0)
      BP=((DEGF*(SAM-1.0))-2.0)*(SQRT(EP/(EP-4.0))-1.0)
      BP=BP*0.5
      VP=(BP/(DEGF-1.))*(HP/CONKP)*(1./((2.*BP)+(DEGF*(SAM-1.))-2.0))
      CP=CON3(DEGF,SAM,LAM1,VP)
      T3=BP/2.0
      Y1=1.0/(1.0+((BP/(DEGF*(SAM-1.0)))*((FO+ CP  )/VP))       )
      PROB=BETINC(0,T1,T3,Y1)
      A4=1.0-BETA
      IF(PROB .GE. A4)  GO TO 11
10    CONTINUE
11    WRITE(IRITE,706)  SAM
706   FORMAT(//,18X,"REQUIRED SAMPLE SIZE IS",F8.1)
      GO TO 13
C
C
C
C ***************************************************************************
C
C
C                  THIS PART OF THE PROGRAM WILL CALCULATE A LAM1 FOR A
C                            GIVEN ALPHA,BETA, AND FIXED SAMPLE SIZE N
C
C
C ***************************************************************************
C
C
C
21    T1=0.5*DEGF*(SAM-1.0)
      T2=0.5*(DEGF-1.0)
      YO=BETINC(1,T1,T2,ALPHA)
      FO=(DEGF*(SAM-1.0)*(1.0-YO))/((DEGF-1.0)*YO)
      T1=(DEGF*(SAM-1.0))/2.0
      DO 625 LM=1,200
      ALTLAM=0.1*LM
      HP=H1(DEGF,SAM,ALTLAM)+H2(DEGF,SAM,ALTLAM)+H3(DEGF,SAM,ALTLAM)
      CONKP=CON2(DEGF,SAM,ALTLAM)
      EP=(HP**2.0)/(CONKP**3.0)
      BP=((DEGF*(SAM-1.0))-2.0)*(SQRT(EP/(EP-4.0))-1.0)
      BP=BP*0.5
      VP=(BP/(DEGF-1.))*(HP/CONKP)*(1./((2.*BP)+(DEGF*(SAM-1.))-2.0))
      CP=CON3(DEGF,SAM,ALTLAM,VP)
      T3=BP/2.0
      Y1=1.0/(1.0+((BP/(DEGF*(SAM-1.0)))*((FO+ CP  )/VP))       )
      PROB=BETINC(0,T1,T3,Y1)
      A4=1.0-BETA
      IF(PROB  .GE. A4)  GO TO 27
625   CONTINUE
27    LAM1=ALTLAM
      WRITE(IRITE,704)  LAMO,LAM1
      WRITE(IRITE,705)  ALPHA,BETA
      WRITE(IRITE,706)  SAM
13    CONTINUE
```

```
C ***************************************************************************
C
C                  THIS PART OF THE PROGRAM CALCULATES THE OC FUNCTION
C                              FOR THE FIXED SIZE TEST
C
C ***************************************************************************
C
C
      WRITE(IRITE,707)
707   FORMAT(///,20X,"OC FUNCTION FOR THE TEST")
      WRITE(IRITE,708)
708   FORMAT(///,14X,"LAMDA",10X,"PROB OF ACCEPTING HO")
      DO 401  IPOW=1,10
      APLAM=LAM0+(((LAM1-LAM0)/9.0)*FLOAT(IPOW-1))
      IF(APLAM>0.0)  GO TO 403
      ANOC=BETINC(0,T1,T2,Y0)
      ANOC=1.0-ANOC
      GO TO 402
403     HP=H1(DEGF,SAM,APLAM)+H2(DEGF,SAM,APLAM)+H3(DEGF,SAM,APLAM)
      CONKP=CON2(DEGF,SAM,APLAM)
      EP=(HP**2.0)/(CONKP**3.0)
      BP=((DEGF*(SAM-1.0))-2.0)*(SQRT(EP/(EP-4.0))-1.0)
      BP=BP*0.5
      VP=(BP/(DEGF-1.))*(HP/CONKP)*(1./((2.*BP)+(DEGF*(SAM-1.))-2.0))
      CP=CON3(DEGF,SAM,APLAM,VP)
      T3=BP/2.0
      Y1=1.0/(1.0+((BP/(DEGF*(SAM-1.0)))*((F0* CP  )/VP))       )
      ANOC=BETINC(0,T1,T3,Y1)
      ANOC=1.0-ANOC
402   WRITE(IRITE,709)  APLAM,ANOC
709   FORMAT(/, 14X,F5.2,17X,F6.4)
401   CONTINUE
      WRITE(IRITE,710)  F0
710   FORMAT(///,16X,"CRITICAL VALUE OF F =",F7.2)
      CVV=(F0*(DEGF-1.0))/(DEGF*(SAM-1.0))
      WRITE(IRITE,711) CVV
711   FORMAT(//,16X,"CRITICAL VALUE OF V =",F10.5)
      READ(IREAD,101)  IREG
101   FORMAT(I2)
      IF(IREG=0)  GO TO 160
C ***************************************************************************
C
C
C                  THIS PART OF THE PROGRAM WILL FIND WALD REGIONS FOR
C                  EVERY N≤NO (THE FIXED SIXE TEST) BY INTERPOLATION
C                  OF THE GHOSH AND WEST TABLES
C
C
C ***************************************************************************
30    CONTINUE
      WRITE(IRITE,715)
715   FORMAT(1H1,20X,"SEQUENTIAL ANOVA TEST")
      WRITE(IRITE,702)
      WRITE(IRITE,703)  DEGF
      WRITE(IRITE,704)  LAM0,LAM1
      WRITE(IRITE,705)  ALPHA,BETA
      WRITE(IRITE,716)
716   FORMAT(///,20X,"THE WALD REGIONS ARE")
      WRITE(IRITE,717)
717   FORMAT(//,10X,"STEP",10X,"LOWER VN", 10X,"UPPER VN")
      NSAM=IFIX(SAM)
306   READ(IREAD,25)  I,AL1,AL2
25    FORMAT(I3,2F10.5)
```

```
      IF(I.LE. NSAM) GO TO 35
      ICOUNT=ICOUNT+1
      IF(ICOUNT>2)  GO TO 40
35    REG(I,1)=AL1
      REG(I,2)=AL2
      GO TO 306
40    DO 150  INDX=1,2
      N1SS=1
      N2SS=1
      N3SS=1
41    IF(REG(N1SS,INDX)>0)  GO TO 42
      N1SS=N1SS+1
      GO TO 41
42    N2SS=N2SS+1
43    IF(REG(N2SS,INDX)>0)  GO TO 44
      N2SS=N2SS+1
      GO TO 43
44    IF(N2SS>N1SS+1)  GO TO 46
      N1SS=N2SS
      GO TO 42
46    N3SS=N2SS+1
47    IF(REG(N3SS,INDX)>0)  GO TO 48
      N3SS=N3SS+1
      GO TO 47
48    L1=N1SS+1
      L2=N2SS-1
      IF(L2>NSAM)  L2=NSAM
      S1=FLOAT(N1SS)
      S2=FLOAT(N2SS)
      S3=FLOAT(N3SS)
      AN=S1+S2+S3
      DO 140  INBT=L1,L2
      SB=FLOAT(INBT)
      Z1=(((AN/SB)-(AN/S2))*((AN/SB)-(AN/S3)))/(((AN/S1)-(AN/S2))
     1*((AN/S1)-(AN/S3)))
      Z2=(((AN/SB)-(AN/S1))*((AN/SB)-(AN/S3)))/(((AN/S2)-(AN/S1))
     2 *((AN/S2)-(AN/S3)))
      Z3=(((AN/SB)-(AN/S1))*((AN/SB)-(AN/S2)))/(((AN/S3)-(AN/S1))
     3 *((AN/S3)-(AN/S2)))
      REG(INBT,INDX)=Z1*REG(N1SS,INDX)+Z2*REG(N2SS,INDX)
     4    +Z3*REG(N3SS,INDX)
140   CONTINUE
      IF( L2=NSAM)  GO TO 150
      N1SS=N2SS
      GO TO 42
150   CONTINUE
155   DO 156 JMF=1,NSAM
      IF(REG(JMF,1)=0.0)  REG(JMF,1)=99999.
      IF(REG(JMF,2)=0.0)  REG(JMF,2)=99999.
      WRITE(IRITE,301) JMF,REG(JMF,1),REG(JMF,2)
156   CONTINUE
      WRITE(IRITE,721)
721   FORMAT(//,80X,"INTERPOLATED")
      GO TO 445
C
C
C  ***************************************************************************
C                  THIS PART OF THE PROGRAM WILL CALCULATE REGIONS FOR
C                     TESTS NOT CONTAINED IN THE GHOSH + WEST TABLES
C
C
C ***************************************************************************
```

```
160   W2=(DEGF-1.0)/2.0
      NSAM=IFIX(SAM)
      WRITE(IRITE,715)
      WRITE(IRITE,702)
      WRITE(IRITE,703)  DEGF
      WRITE(IRITE,704)  LAMO,LAM1
      WRITE(IRITE,705)  ALPHA,BETA
                                                                    SE
                                                                  START OF
      WRITE(IRITE,716)
      WRITE(IRITE,717)
      BOUND(1)=     ALOG(BETA/(1.0-ALPHA))
      BOUND(2)=      ALOG((1.0-BETA)/ALPHA)
      XLIN(1)=  (2.0*BOUND(1)+ (LAM1-LAMO))/(-2.0*BOUND(1))
      XLIN(2)= (2.0*BOUND(2)+(LAM1-LAMO))/(-2.0*BOUND(2))
      DO 220  NSZ=2,NSAM
      ECON=-(FLOAT(NSZ)*(LAM1-LAMO))/2.0
      W1=((DEGF*FLOAT(NSZ))-1.0)/2.0
      ZCON1=(FLOAT(NSZ)*LAM1)/2.0
      ZCONO=(FLOAT(NSZ)*LAMO)/2.0
      DO 210 IB=1,2
      XEVAL=0.0
      IF( XLIN(IB)  .GT. 0.0  .AND. XLIN(IB)  .GT. 1.0)  GO TO 297
      IF(XLIN(IB)  .GT. 0.0)  GO TO 170
      VALO=  EXP(ECON) - EXP(BOUND(IB))
      DO 296 ISR=1,9
      XSR=  -(ISR*0.1)
      SEAR=  ( XSR/(1.0+XSR))
      W3=ZCON1*SEAR
      W4=ZCONO*SEAR
      HYP1=  CONHYP(W1,W2,W3)
      HYP2=  CONHYP(W1,W2,W4)
      VAL2 =(  EXP(ECON)*HYP1) -(  EXP(BOUND(IB))*HYP2)
      IF( (VALO*VAL2)  .LE.  0.0  .AND. XLIN(IB)  .LT. 0.0) GO TO 210
296   CONTINUE
      SEAR=0.99
      W3= ZCON1*SEAR
      W4=ZCONO*SEAR
      HYP1=     CONHYP(W1,W2,W3)
      HYP2=       CONHYP(W1,W2,W4)
      VAL2= (   EXP(ECON)*HYP1) -(  EXP( BOUND(IB))*HYP2)
      IF( (VALO*VAL2)  .GT. 0.0)  GO TO 210
297   W3=ZCON1*0.5
      W4=ZCONO*0.5
      HYP1=  CONHYP(W1,W2,W3)
      HYP2=   CONHYP(W1,W2,W4)
      VALST=(  EXP(ECON)*HYP1) -(  EXP(BOUND(IB))*HYP2)
      IF( (VALST*VALO) .GT. 0.0)  GO TO 197
      XLIN(IB)=1.0
      GO TO 170
197   DO 386 IFND=10,60 ,10
      SEARS=IFND/(1.0+IFND)
      W3=ZCON1*SEARS
      W4=ZCONO*SEARS
      HYP1=  CONHYP(W1,W2,W3)
      HYP2=   CONHYP(W1,W2,W4)
      VALS=  (  EXP(ECON)*HYP1) -( EXP(BOUND(IB))*HYP2)
      IF((VALST*VALS)) 387,386,386
387   XLIN(IB)=IFND-5.0
      GO TO 170
386   CONTINUE
      XLIN(IB)=IFND
```

```
170   CONTINUE
      IF(XLIN(IB)  .GT. 0.0)  GO TO  195
      XLIN(IB)= XLIN(IB)+XLN
      IF( XLIN(IB)  .EQ. -1.0)  XLIN(IB)=-0.5
      GO TO 170
195   W3= ZCON1*(XLIN(IB)/(1.0+XLIN(IB)))
      W4=ZCON0*(XLIN(IB)/(1.0+XLIN(IB))  )
      HYP1=CONHYP(W1,W2,W3)
      HYP2=CONHYP(W1,W2,W4)
      Y1=W1+1.0
      Y2=W2+1.0
      HYP3=CONHYP(Y1,Y2,W3)
      HYP4=CONHYP(Y1,Y2,W4)
      AUX1=(FLOAT(NSZ)*W1)/(2.0*W2*((1.0+XLIN(IB))**2.0))
      AUX2=((LAM1*HYP3)/HYP1)-((LAM0*HYP4)/HYP2)
      FPX=AUX1*AUX2
      HYP1=  DLOG(HYP1)
      HYP2=   DLOG(HYP2)
      FX=ECON+HYP1-HYP2-BOUND(IB)
      IF(  DABS(FPX)  .LE. 2.0)  GO TO 501
503   XLN=  XLIN(IB)-(FX/FPX)
      IF(  DABS(XLIN(IB)-XLN)  .LE. EPS  ) GO TO 210
      XINT=XLIN(IB)
      XLIN(IB)=XLN
      XLN=XINT
      XEVAL=FX
      GO TO 170
501   IF(  (XEVAL*FX)) 502,503,503
502   XNXT=  (  XLIN(IB)+XLN)*0.5
      W3=ZCON1*(XNXT/(1.0+XNXT))
      W4= ZCON0*(XNXT/(1.0+XNXT))
      HYP1=  CONHYP( W1,W2,W3)
      HYP2=   CONHYP(W1,W2,W4)
      FXNXT=  ECON+ DLOG(HYP1)- DLOG(HYP2)  -BOUND(IB)
      IF((XEVAL*FXNXT)) 505,210,506
505   XLIN(IB)=XNXT
      FX=FXNXT
      GO TO 507
506   XLN=XNXT
      XEVAL=FXNXT
507   IF(  DABS(XLIN(IB)-XLN)  .LE. EPS)  GO TO 210
      GO TO 502
210   CONTINUE
      WRITE(IRITE,301) NSZ,XLIN(1),XLIN(2)
301   FORMAT(11X,I2,10X,F8.4,10X,F8.4)

220   CONTINUE
      WRITE(IRITE,720)
720   FORMAT(//,80X,"CALCULATED")

445    CONTINUE
      STOP
      END
```

START OF SE

```
      FUNCTION ZI(X,A,B)
      FN=.7*(ALOG(15.+A+B))**2+AMAX1(X*(A+B)-A,0.0)
      N=INT(FN)
      C=1.-(A+B)*X/(A+2.*FN)
      ZI=2./(C+SQRT(C**2-4.*FN*(FN-B)*X/(A+2.*FN)**2))
      DO 60 J=1,N
      FN=N+1-J
      A2N=A+2.*FN
      ZI=(A2N-2.)*(A2N-1.-FN*(FN-B)*X*ZI/A2N)
      ZI=1./(1.-(A+FN-1.)*(A+FN-1.+B)*X/ZI)
60    CONTINUE
      RETURN
      END
```

SEG

```
      FUNCTION CGAM(A)
      AA=A
      CAC=0.0
      IF(A-2.)2,8,8
2     IF(A-1.)4,6,6
4     CAC=-2.+(A+.5)*ALOG(1.+1./A)+(A+1.5)*ALOG(1.+1./(A+1.))
      AA=A+2.
      GO TO 8
6     CAC=-1.+(A+.5)*ALOG(1.+1./A)
      AA=A+1.
8     CA=2.269489/AA
      CA=.52560647/(AA+1.0115231/(AA+1.5174737/(AA+CA)))
      CA=.0833333333/(AA+.03333333/(AA+.25238095/(AA+CA)))
      CGAM=CA+CAC
      RETURN
      END
```

```
                                                                    START OF SE
      FUNCTION BETINC(IND,A,B,X)
C     INCOMPLETE BETA FUNCTION AND ITS INVERSE
C    MARK=1 FOR INVERSE (SEND DOWN PROB)
      CAB=CGAM(A+B)-CGAM(A)-CGAM(B)+.5*ALOG((A+B)*6.28318531)
      IF(IND)10,10,20
10    EP=CAB+A*ALOG(X*(1.+B/A))+B*ALOG((1.-X)*(1.+A/B))
       IF(X-A/(A+B))12,12,14
12    BETINC=ZI(X,A,B)*EXP(EP+.5*ALOG(B/A))
      RETURN
14    BETINC=1.-ZI(1.-X,B,A)*EXP(EP+.5*ALOG(A/B))
      RETURN
20    IF(X-.5)22,22,24
22    QZ=ALOG(X)
      IGO=1
      AA=A
      BB=B
      GO TO 26
24    QZ=ALOG(1.-X)
      IGO=2
      AA=B
      BB=A
26    XT=AA/(AA+BB)
      CABB=CAB+.5*ALOG(BB/AA)+AA*ALOG(1.+BB/AA)+BB*ALOG(1.+AA/BB)
      DO 40 NC=1,100
      ZZ=ZI(XT,AA,BB)
      QX=CABB+AA*ALOG(XT)+BB*ALOG(1.-XT)+ALOG(ZZ)
      XC=(QZ-QX)*(1.-XT)*ZZ/AA
      XC=AMAX1(XC,-.99)
      XC=AMIN1(XC,.5/XT-.5)
      XT=XT*(1.+XC)
      IF(ABS(XC)-1.E-6)42,40,40
40    CONTINUE
42    GO TO (44,46),IGO
44    BETINC=XT
      RETURN
46    BETINC=1.-XT
      RETURN
      END
```

```
      FUNCTION CONHYP(XF,YF,UF)
      COMMON  EPS
      DOUBLE PRECISION   TSUM
      TAU(AR)=ALGAMA(AR)
      X=XF
      Y=YF
      U=UF
      PMULT=1.0
      TSUM=1.0
      IF(X*Y)  101,100,101
100   CONHYP=  EXP(U)
      RETURN
101   IF(U)  103,102,104
102   CONHYP=  1.00
      RETURN
103   X=Y-X
      PMULT=  EXP(U)
      U= -U
104   IF(X)  105,102,106
105   ICHK=-IFIX(X)
      TEST=ICHK+X
      IF(TEST) 111,108 ,111
108   IF( ICHK-1)  111,107,109
107   CONHYP=(1.0-(U/Y))*PMULT
      RETURN
109   INDX=ICHK
      XSTAR=1.0
      DO 110  N=1,INDX
      XSTAR= XSTAR*( X+N-1.0)
      T1= Y+N
      T2=  FLOAT(N+1)
      T3=  FLOAT(N)
      YSTAR= (TAU(Y)-TAU(T1)-TAU(T2))+(T3*  ALOG(U))
      TSUM= TSUM+ (XSTAR*  EXP(YSTAR))
110   CONTINUE
      CONHYP=  PMULT*TSUM
      RETURN
111   DO 125  IT=1,50
      T=FLOAT(IT)
      T1=T+X
      T2=T+Y
      T3=T+1.0
      PS= (  GAMMA(Y)  / GAMMA(X))*(  GAMMA(T1) /GAMMA(T2))
      PF=  (T*  ALOG(U))- TAU(T3)
      PS=PS *  EXP(PF)
      TSUM=  TSUM+PS
      IF(  ABS(PS)  .LE. EPS)  GO TO 112
125   CONTINUE
112   CONHYP=  TSUM* PMULT
      RETURN
106   DO 115  IT=1,50
      T= FLOAT(IT)
      T1=T+X
      T2=T+Y
      T3=T+1.0
      PS=TAU(Y)+TAU(T1)-TAU(X)-TAU(T2)-TAU(T3)
      PS=  EXP(PS)*(U**T)
      TSUM=  TSUM+PS
      IF (  ABS(PS)  .LE.  EPS)  GO TO 120
115   CONTINUE
120   CONHYP=  TSUM* PMULT
      RETURN
```

FIXED SAMPLE ANOVA TEST

****

K=2.0 GROUPS

HO:LAMO = 0.00 VS HI:LAM1 = 1.00

ALPHA = 0.01 BETA = 0.01

REQUIRED SAMPLE SIZE IS 27.0

OC FUNCTION FOR THE TEST

LAMDA	PROB OF ACCEPTING HO
0.00	0.9900
0.11	0.8180
0.22	0.5793
0.33	0.3673
0.44	0.2154
0.56	0.1192
0.67	0.0631
0.78	0.0323
0.89	0.0160
1.00	0.0078

CRITICAL VALUE OF F = 7.15

CRITICAL VALUE OF V = 0.13748

SEQUENTIAL ANOVA TEST

****

K=2.0 GROUPS

H0:LAM0 = 0.00 VS H1:LAM1 = 1.00

ALPHA = 0.01 BETA = 0.01

THE WALD REGIONS ARE

STEP	LOWER VN	UPPER VN
2	-0.8912	-1.1088
3	-0.8912	-1.1088
4	-0.8912	-1.1088
5	-0.8912	42.7986
6	-0.8912	3.5556
7	-0.8912	1.8734
8	-0.8912	1.2850
9	-0.8912	0.9872
10	0.0049	0.8082
11	0.0104	0.6891
12	0.0156	0.6044
13	0.0204	0.5412
14	0.0250	0.4924
15	0.0293	0.4535
16	0.0333	0.4219
17	0.0371	0.3956
18	0.0406	0.3736
19	0.0438	0.3548
20	0.0468	0.3385
21	0.0497	0.3244
22	0.0523	0.3120
23	0.0548	0.3010
24	0.0571	0.2912
25	0.0593	0.2824
26	0.0614	0.2745
27	0.0633	0.2674

APPENDIX C

A COMPUTER PROGRAM FOR k=2 SANOVA

Chapter 2 of this thesis derived a procedure for obtaining the properties of a k=2 SANOVA test. The procedure has been summarized in Figures 12 and 13. As previously discussed, this procedure cannot feasibly be performed analytically. Section (2.6) considered an alternative, a numerical implementation of the theory. This appendix contains a computer program for obtaining the properties of a k=2 SANOVA test utilizing the approach discussed in Section (2.6).

The program is written in Fortran IV for use on a Burroughs or CDC computer. Its implementation on other machines may require modifications, specifically the statements involving a read or write from disc.

To use the program, the user must supply the following information:

1) λ_0, λ_1, k, m_0

   On one card in a (3F10.5,I3) format.

   Note k is the number of means and should always be input as 2; m_0 is the truncation point.

2) V_A^i, V_R^i, $i=1, \cdots, m_0$.

Two numbers per card (V_A^i being the acceptance region and V_R^i the rejection) in a (2F15.0) format. Any time acceptance is not possible at a given stage j, V_A^j should be input as a negative number. Similarly, any time rejection is not possible, V_R^j should be input as a number greater than 10^{10}. Note that the first region card should always be of the form $V_A^1 = -1$, $V_R^1 = 10^{10}$, since no decision can be made at this stage.

3) Gridsize in an (F10.0) format.

This represents the coarseness of the grid or the quantities h_Q and h_W of Section (2.6). The program assumes that $h_Q = h_W$. It is best to select this number as a power of 2, e.g., 0.5, 0.25, etc. In general, the smaller this number, the more accurate the results but the larger the amount of computation required. As discussed in Section (2.6), the most efficient approach is to perform several runs, using a finer grid size on each run.

The program uses two random access disc files (files 1 and 2). These files represent the density at stages i and i+1($i=n_1,\cdots, m_0-1$). The first file is used to compute the second file as discussed in Section (2.6). The size of these files is dependent upon the choice of the gridsize parameter. However, 125,000 words per file should be sufficient for most problems.

The program output consists of the probability of acceptance, rejection and continuation (P_A^i, P_R^i, P_C^i) for each stage for every value of λ. These quantities are then used to compute summary OC and ASN curves.

Currently, the program is being implemented on a CDC computer by Mr. Kent Kaufmann of Western Illinois University. The program will be used to generate a brief set of OC and ASN curves for several k=2 SANOVA tests.

```
      FILE   1=ST1/MIL,UNIT=DISK,SAVE=99,LOCK,RANDOM,BLOCKING=30,RECORD=1
      FILE   2=ST2/MIL,UNIT=DISK,SAVE=99,LOCK,RANDOM,BLOCKING=30,RECORD=1
      FILE   10=RES/MIL,UNIT=DISK,SAVE=99,LOCK,RANDOM,RECORD=32
      FILE   11=CUR/MIL,UNIT=DISK,SAVE=99,LOCK,RANDOM,BLOCKING=5,RECORD=6
      FILE   5=CARD,UNIT=READER                                          00000500
      FILE   6=OUTPUT,UNIT=PRINTER                                       00000600
      FILE   7=DEBUG,UNIT=PRINTER                                        00000700
      FILE   8=TEMP,UNIT=PRINTER                                         00000800
      C                                                                  00000900
      C                                                                  00001000
      C                                                                  00001100
      C                  SEQUNTIAL ANALYSIS OF VARIANCE                  00001200
      C                       RW MILLER                                  00001300
      C                                                                  00001400
      C        THIS PROGRAM CALCULATES THE AVERAGE SAMPLE NUMBER.        00001500
      C        MEDIAN SAMPLE SIZE,AND OPERATING CHARACTERISTIC FUNCTION FOR 00001600
      C        A SEQUENTIAL TEST OF THE EQUALITY OF K MEANS              00001700
      C        THE TEST IS CHARACTERIZED BY A LAMO,LAM1,AND REGIONS      00001800
      C                                                                  00001900
      C                                                                  00002000
      C                                                                  00002100
                                                          START OF SEGMENT
            COMMON /CB10/CA,PA,CR,PR,A,B,C                               00002200
            COMMON /CB1/GRIDW,GRIDQ,GRIDR                                00002300
            COMMON /CB2/ JREC,TSTAT                                      00002400
            COMMON /CB3/ NCONW,NCONQ,NSTRT                               00002500
            COMMON /CB5/ REG(30,2)                                       00002600
            COMMON /CB4/ DEGF,ALAM                                       00002700
            COMMON /CB6/ TMAXW,TMINW,TMAXQ,TMINQ                         00002800
            COMMON /CB8/ JOINT,ICAL                                      00002900
            COMMON /CB11/ SINE45                                         00003000
            COMMON /CB12/ NUMW,NUMQ,NUMR                                 00003100
            COMMON / CB14/ XMEAN(2),XBR1,XBR2,VAR,DGF                    00003200
            COMMON /CB7/ LSTP,ISUR                                       00003300
            COMMON /CB15/ GREGC(14,14)                                   00003400
            COMMON /CB20/ RECMAX
            COMMON/ERR/ IRP1,IRP2                                        00003500
            REAL LAMO,LAM1                                               00003600
      C
      C
      C        THIS IS NECCESSARY FOR START=RESTART
      C
            COMMON/RESTAR/OC(30,2),ASN(30),NTESTS
            COMMON /RESTAR/ KTEST,NOC,NSTP,I1,I2,I3,KREC,IRAC,NPFIAC,IRNR
            COMMON /RESTAR/ NPFINR,PROBAC,PROBNR,PRRAC,PRQAC,PRRNR,PRQNR
            COMMON /RESTAR/ RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR
            COMMON /RESTAR/ WN,QN,RN
            EQUIVALENCE (RNBEG,RNRBEG)
      C
      C
            ELIPFI(Q,W,R,N,VR)=(((Q*COS(45.)-W*SIN(45.))**2.)/(R*FLOAT(N)+RAD)00003800
           1  )+(((Q*SIN(45.)+W*COS(45.))**2.)/((R*FLOAT(N)*VR)/(2.*(FLOAT(N)00003900
           2 **4.0)*VR))*RAD))-1.0                                       00004000
            PARAR(XC,XP,QC,WC)=(XC*((QC-(XP*WC/XC))**2.))+(XC*(WC**2.))  00004100
           1     -(((XP*WC)**2.)/XC)                                     00004200
            DATA IREAD,IRITE/5,6/                                        00004300
            EPS= 1.0E-6                                                  00004400
            NPFI=6                                                       00004500
            SINE45=SIN(0.78539816)                                       00004600
            READ(10=1) NTRYS,ITEST,KOC,MSTP
            NTRYS=NTRYS+1
            READ(IREAD,199) NTESTS                                       00004700
```

```
C                                                                       0000490
C       INPUT TO THIS PROGRAM CONSISTS OF LAM0,LAM1,K=NUMBER OF MEANS   0000500
C       AND THE TRUNCATION POINT   AS WELL AS THE REGIONS               0000510
C                                                                       0000520
      DO 9112 KTEST=1,NTESTS                                            0000530
      READ(IREAD,205)  LAM0,LAM1,DEGF,MTP                               0000540
205   FORMAT(3F10.5,I3)                                                 0000550
C                                                                       0000560
C       REG(I,1)=VALUE OF V1 FOR WHICH ANY V≤V1 RESULTS IN ACCEPTANCE   0000570
C       REG(I,2)=VALUE OF V1 FOR WHICH ANY V≥V1 RESULTS IN REJECTION    0000580
C                                                                       0000590
C                                                                       0000600
C       IF IT IS NOT POSSIBLE TO ACCEPT AT STEP I REG(I,1)=0.0          0000610
C       IF IT IS NOT POSSIBLE TO REJECT AT STEP I REG(I,2)=1.E10        0000620
C                                                                       0000630
      READ (IREAD,206) ((REG(J1,1),REG(J1,2)),J1=1,MTP)                 0000640
206   FORMAT(2F15.0)                                                    0000650
      NUM=IFIX(DEGF)-1                                                  0000660
      DO 20  M=1,NUM                                                    0000670
      XMEAN(M)=0.0                                                      0000680
20    CONTINUE                                                          0000690
      NUM=NUM+1                                                         0000700
C                                                                       0000710
C                                                                       0000720
C       A SEARCH IS MADE TO DETERMINE THE FIRST STEP AT                 0000730
C       WHICH A DECISION IS POSSIBLE                                    0000740
C                                                                       0000750
      DO 35 NSTP=1,MTP                                                  0000760
      IF( REG(NSTP,1) .LE. 0.0 .AND. REG(NSTP,2) .GE. 1.E10) GO TO 25   0000780
      ISUR=NSTP                                                         0000790
      NCAL=ISUR+1                                                       0000800
      GO TO 40                                                          0000810
25    CALL FSTEPPROB(OC(NSTP,1),OC(NSTP,2),ASN(NSTP),NSTP)              0000770
      IF( KTEST .GT. ITEST .OR. NOC .GT. KOC)
     1WRITE(11=(NSTP+((NOC-1)*NTESTS)+(KTEST-1)*NTESTS*10)) OC(NSTP,1),
     2 OC(NSTP,2),ASN(NSTP),NSTP,ALAM,KTEST
35    CONTINUE                                                          0000620
40    READ(IREAD,207)  GRIDSZ                                           0000830
207   FORMAT(F10.0)                                                     0000840
      IF(ITEST .GT. KTEST) GO TO 9112
      GRIDW=GRIDSZ                                                      0000850
      GRIDQ=GRIDSZ                                                      0000860
      GRIDR=GRIDW**2.0                                                  0000870
      RAD=(3.0*(GRIDW**2.0+GRIDQ**2.0))**0.5                            0000880
C                                                                       0000890
C       THE DIMMENSIONS OF THE GRID ARE CALCULATED                      0000900
C                                                                       0000910
      XVAL= 10.0*(((0.4)**DEGF)**0.5)                                   0000920
      MAXVAL=XVAL*((2.0*DEGF*FLOAT(ISUR-1))**0.5)+(DEGF*(FLOAT(ISUR-1)))0000930
      STORE= MAXVAL/GRIDR                                               0000940
      NUMR=IFIX(STORE)+1                                                0000950
      VAR= 4.0*(FLOAT(ISUR)**0.5)                                       0000960
      NUMW=((IFIX(VAR/GRIDW)+1)*2)+1                                    0000970
      NUMQ=((IFIX(VAR/GRIDQ)+1)*2)+1                                    0000980
      RECMAX=NUMQ*NUMR*NUMW                                             0001000
      VAR= (FLOAT(ISUR)**0.5)                                           0001010
      DGF=DEGF*FLOAT(ISUR-1)                                            0001020
      NSTRT=ISUR                                                        0001021
      WRITE(IRITE,1401)                                                 0001030
1401  FORMAT(1H1,10X,"SEQUENTIAL ANALYSIS OF VARIANCE")                 0001040
      WRITE(IRITE,1402)                                                 0001050
1402  FORMAT(//,20X,"THE TEST IS")                                      0001060
      WRITE(IRITE,1403) LAM0,LAM1                                       0001070
```

```
1404  FORMAT(//,15X,"WITH K =",F2.0)                                    00011000
      WRITE(IWRITE,1405)                                                00011100
1405  FORMAT(//,15X,"AND THE FOLLOWING REGIONS")                        00011200
      WRITE(IWRITE,1406)                                                00011300
1406  FORMAT(12X,"STEP",5X,"VN ACCEPT",5X,"VN REJECT")                  00011400
      DO 1409 IAM=1,MTP                                                 00011500
      IF(REG(IAM,1) .LE. 0.0) REG(IAM,1)=-9999.99                       00011600
      WRITE(IWRITE,1407) IAM,REG(IAM,1),REG(IAM,2)                      00011700
1407  FORMAT(13X,I2,6X,F8.4,6X,F8.4)                                    00011800
1409  CONTINUE                                                          00011900
      WRITE(IWRITE,1410) GRIDW,GRIDQ,GRIDR                              00012000
1410  FORMAT(//,12X,"GRIDW=",F6.3,3X,"GRIDQ=",F6.3,3X,"GRIDR=",F6.3)    00012100
      WRITE(IWRITE,1411) NUMW,NUMQ,NUMR                                 00012200
1411  FORMAT(12X,"SIZEW=",I4,5X,"SIZEQ=",I4,5X,"SIZER=",I4)             00012300
      WRITE(IWRITE,1412) RECMAX                                         00012400
1412  FORMAT(/,12X,"TOTAL NUMBER OF GRID POINTS USED=",I9)              00012500
      CALL BOUR(1,1,1,0,WN,QN,RN)                                       00012600
      PRB1=PHI(WN,0,VAR)                                                00012700
      VALUE=AMAX1(GRIDR,POSPROB(WN,QN,RN,ISUR))                         00012800
      PRB2=CHISQ(VALUE,DGF)                                             00012900
      CALL BOUR(1,1,NUMR,0,WN,QN,RN)                                    00013000
      PRB3=CHISQ((POSPROB(WN,QN,RN,ISUR)),DGF)                          00013100
      WRITE(IWRITE,1413) PRB1,PRB2,PRB3                                 00013200
1413  FORMAT(12X,"MIN W PROB =",E15.5,/,12X,"RANGE OF R PROB =",E15.5,  00013300
     1 5X,E15.5)                                                        00013400
      WRITE(IWRITE,202)                                                 00013500
202   FORMAT(1H1,30X,"SEQUENTIAL ANOVA")                                00013600
      WRITE(IWRITE,203)                                                 00013700
203   FORMAT(///,5X,"LAMBDA",10X,"MSN",10X,"ASN",10X,"OC",10X,"POW")    00013800
C                                                                       00014000
C       SEVERAL POINTS ON THE OC CURVE                                  00014100
C       ARE CALCULATED FOR A SEQUENTIAL TEST                            00014200
C             WITH THESE REGIONS                                        00014300
C                                                                       00014400
                                                                        SEGMENT
      DO 500 NOC=1,10,9                                                 00014500
      IF( NOC .GT. NOC) GO TO 500
      ALAM=LAM0+ ((LAM1-LAM0)/9.0)*FLOAT(NOC-1)                         00014600
      WRITE(7,204) ALAM                                                 00014700
204   FORMAT(1H1,30X,"LAMBDA=  ",F6.3)                                  00014800
                                                                        00014900
C                                                                       00015000
C       FOR A GIVEN LAMDA VALUES OF MEAN1 AND MEAN2 CAN BE              00015100
C       CALCULATED. THESE ARE NEEDED FOR THE BOOKING                    00015200
C       SUBROUTINE AND TO CALCULATE CERTAIN PROBABLITIES                00015300
C       SUCH AS THE F(W,Q,R) AT THE FIRST STEP                          00015400
C       AND FOR PHI(Z) AND PHI(0)                                       00015500
C                                                                       00015600
                                                                        00015700
      XMEAN(NUM1)=  SQRT((DEGF*ALAM)/(DEGF-1.0))                        00015800
      XBR1= XMEAN(1)*FLOAT(ISUR)                                        00015900
      XBR2=XMEAN(2)*FLOAT(ISUR)                                         00016000
      NCONW=IFIX((ISUR*XMEAN(1))/GRIDW)-IFIX(NUMW/2.)                   00016100
      NCONQ=IFIX((ISUR*XMEAN(2))/GRIDQ)-IFIX(NUMQ/2.)                   00016200
      JOINT=1                                                           00016300
      ICAL=2                                                            00016400
      CALL BOUR(1,1,1,0,TMINW,TMINQ,TMINR)                              00016500
      CALL BOUR(NUMW,NUMQ,NUMR,0,TMAXW,TMAXQ,TMAXR)                     00016600
      WRITE(7,972) XMEAN(1),XMEAN(2),VAR,XBR1,XBR2                      00016700
972   FORMAT(///,5X,"MEAN1=",E15.7,2X,"MEAN2=",E15.7,2X,"VAR=",E15.7,2X,00016800
     1"SMEAN1=",E15.7,2X,"SMEAN2=",E15.7)                               00016900
      WRITE(7,801)  NUMR,NUMQ,NUMW                                      00017000
801   FORMAT(20X,3I7)                                                   00017100
      WRITE(7,802) TMINQ,TMINW,TMAXQ,TMAXW,TMAXR                        00017200
                                                                        00017300
```

```
241   FORMAT(///,8X,"STEP",10X,"PROB. ACCEPT",13X,"PROB. REJECT",11X,     00017500
     1"PROB. CONTINUE")                                                   00017600
      CALL FSTEPPROB( OC(ISUR,1),OC(ISUR,2),ASN(ISUR),ISUR)
      IF( KTEST .GT. ITEST .OR. NOC .GT. KOC)
     1WRITE(11=(ISUR+((NOC-1)*NTESTS)+(KTEST-1)*NTESTS*10))OC(ISUR,1),
     2 OC(ISUR,2),ASN(ISUR),ISUR,  ALAM,KTEST
      WRITE(7,208) ISUR,OC(ISUR,1),OC(ISUR,2),ASN(ISUR)                   00017700
      LSTP=ISUR                                                           00017800
      CR=1./ISUR                                                          00017900
      PR=0.0                                                              00018000
      CA=0.0                                                              00018100
C                                                                         00018200
C        THIS PART OF THE PROGRAM CALCULATES                              00018300
C        F(WN,QN,RN) FROM G( W(N-1), Q(N-1), R(N-1), Z, U)                00018400
C   WHERE G(W(N-1),Q(N-1),R(N-1),Z,U)=FT(W(N-1),Q(N-1),R(N-1))*P(Z)*P(U)  00018500
                                                                          00018600
C        IN ORDER TO FIND THE QUANTITIES OF INTEREST IN                   00018700
C        SEQUENTIAL ANALYSIS, NAMELY THE ASN AND OC CURVE                 00018800
C        THE PROBABILITY DISTRIBUTION MUST BE FOUND                       00018900
C        AT EVERY STEP N                                                  00019000
C        THEN THIS DISTRIBUTION CAN BE INTEGRATED TO FIND                 00019100
C        THE PROBABILITIES OF ACCEPTING ,REJECTING AND CONTINUING AT      00019200
C                              EVERY STEP                                 00019300
                                                                          00019400
      WRITE(8,777)
777   FORMAT(1H1)
      DO 400 NSTP=NCAL,MTP                                                00019500
      IF( MSTP .GT. NSTP) GO TO 399
      LSTP=NSTP-1                                                         00019600
      IF( REG(LSTP,1) .LE. 0.0) GO TO 131                                 00019700
      CA=((2.0*REG(LSTP,1))+1.0)/(2.0*LSTP*REG(LSTP,1))                   00019800
      PA=1.0/(2.0*LSTP*REG(LSTP,1))                                       00019900
131   IF( REG(LSTP,2) .GE. 1.E6) GO TO 132                                00020000
      CR=((2.0*REG(LSTP,2))+1.0)/(2.0*LSTP*REG(LSTP,2))                   00020100
      PR=1.0/(2.0*LSTP*REG(LSTP,2))                                       00020200
      GO TO 133                                                           00020300
132   CR=1.0/FLOAT(LSTP)                                                  00020400
      PR=0.0                                                              00020500
133   PROBAC=0.0                                                          00020600
      PROBNR=0.0                                                          00020700
      PRRAC=0.0                                                           00020800
      PRQAC=0.0                                                           00020900
      PRRNR=0.0                                                           00021000
      PRQNR=0.0                                                           00021100
      DO 1130 I1=1,NUMW                                                   00021200
      PRQAC=0.0                                                           00021300
      PRQNR=0.0                                                           00021400
      DO 1131 I2=1,NUMQ                                                   00021500
      CALL BOUR(I1,I2,I,0,WN,QN,RN)                                       00021600
      IRAC=0                                                              00021700
      IRNR=0                                                              00021800
      RVALAC=1.E30                                                        00021900
      IF( CA .LE. 0) GO TO 1121                                           00022000
      RVALAC=PARAR(CA,PA,QN,WN)                                           00022100
1121  RVALNR=PARAR(CR,PR,QN,WN)                                           00022200
      PRRAC=0.0                                                           00022300
      PRRNR=0.0                                                           00022400
      DO 5132 I3=1,NUMR
      IF( NTRYS .GT. 1 .AND. PTIM1 .LE. 0.0)
     1          CALL RESUME( PTIM1,PTIM2,PTIM3,$5132)
      IF( I3 .LE. 1) GO TO 1122                                           00022600
      CALL RCAL(I1,I2,I3,0,WN,QN,RN)                                      00022700
1122  A=WN                                                                00022800
```

```
      KREC=JREC                                                          0002310
      IF( POSPROB(WN,QN,RN,NSTP) .LT. 0.0) GO TO 1132                    0002320
                                                                         0002330
C                                                                        0002340
C        THIS STATEMENT CALCULATES THE DENSITY AT POINT A,B,C            0002350
C        AT NSTP BY INTEGRATING OVER A TWO DIMENSIONAL REGION IN LSTP    0002360
C                                                                        0002370
                                                                         0002380
      PROBTS=DENSTS(A,B,C,KREC)                                          0002390
                                                                         0002400
C                                                                        0002410
C                                                                        0002420
      IF( C .LT. RVALAC) GO TO 1124                                      0002430
      IRAC=IRAC+1                                                        0002440
      IF( IRAC .GT. 1) GO TO 1123                                        0002450
      NPFIAC=NUMR-I3+1                                                   0002460
      RACBEG=C                                                           0002470
      SPROAC=PROBTS                                                      0002480
C                                                                        0002490
C                                                                        0002500
C        THIS PART OF THE PROGRAM CALCULATES                             0002510
C        THE PROBABILITIES OF ACCEPTING,REJECTING,AND CONTINUING         0002520
C                              AT STEP NSTP                              0002530
C        THESE ARE OBTAINED BY PERFORMING A THREEE DIMENSIONAL INTEGRATI 0002540
                                                                         0002550
C        THIS THREE DIMENSIONAL INTEGRATION IS IS DONE NUMERICALLY       0002560
C        BY THREE SUCCESSIVE 1 DIMENSIONAL INTEGRALS                     0002570
C        EACH 1 DIMENSIONAL INTEGRATION IS DONE VIA                      0002580
C        A 14 POINT(IF POSSIBLE) NEWTON-GREGORY FORMULA                  0002590
                                                                         0002600
                                                                         0002610
 1123 PRRAC=PRRAC+WEIGHT(NPFIAC,IRAC)*PROBTS                             0002620
 1124  IF( C .LT. RVALNR) GO TO 1132                                     0002630
      IRNR=IRNR+1                                                        0002640
      IF( IRNR .GT. 1) GO TO 1125                                        0002650
      NPFINR=NUMR-I3+1                                                   0002660
      RNRBEG=C                                                           0002670
      SPRONR=PROBTS                                                      0002680
 1125 PRRNR=PRRNR+WEIGHT(NPFINR,IRNR)*PROBTS                             0002690
 1132 TMEL1=PTIM1+TIME(2)/3600.0
      TMEL2=PTIM2+TIME(3)/3600.0
      TMEL3=PTIM3+TIME(4)/3600.0
      WRITE(10,1)  NTRYS,NTEST,NOC,NSTP,JOINT,ICAL,I1,I2,I3,KREC,IRAC,
     1 NPFIAC,IRNR,NPFINR,A,B,C,PROBAC,PROBNR,PRRAC,PROAC,PRRNR,PRONR,
     2  RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR, TMEL1,TMEL2,TMEL3
 5132 CONTINUE
      IF( IRAC .EQ. 0) GO TO 1126                                        0002710
      Y1=TERPO(A,B,RVALAC,ICAL)                                          0002720
      ADDA=ABS(RVALAC-RACBEG)*0.5*(SPROAC+Y1)                            0002730
      PRRAC=GRIDR*PRRAC+ADDA                                             0002740
      PROAC=PROAC+WEIGHT(NUMQ,I2)*PRRAC                                  0002750
 1126 IF( IRNR .EQ. 0) GO TO 1131                                        0002760
      Y1=TERPO(A,B,RVALNR,ICAL)                                          0002770
      ADDR=ABS(RVALNR-RNRBEG)*0.5*(SPRONR+Y1)                            0002780
      PRRNR=GRIDR*PRRNR+ADDR                                             0002790
      PRONR=PRONR+WEIGHT(NUMQ,I2)*PRRNR                                  0002800
 1131 CONTINUE                                                           0002810
      PRONR=PRONR*GRIDQ                                                  0002820
      PROAC=PROAC*GRIDQ                                                  0002830
      PROBAC=PROBAC+WEIGHT(NUMW,I1)*PROAC                                0002840
      PROBNR=PROBNR+WEIGHT(NUMW,I1)*PRONR                                0002850
 1130 CONTINUE                                                           0002860
      PROBNR=PROBNR*GRIDW                                                0002870
      PROBAC=PROBAC*GRIDW                                                0002880
```

```
      JOINT=ICAL
      ICAL=INTER
                                                                    S
                                                                START OF

C         OC(NSTP,1)=PROBABILITY OF ACCEPTING AT STEP NSTP
C         OC(NSTP,2)=PROBABILITY OF REJECTING AT STEP NSTP
C         ASN(NSTP)=PROBABILITY OF CONTINUING AT STEP NSTP

      ASN(NSTP)=PROBNR-PROBAC
      OC(NSTP,1)=PROBAC
      OC(NSTP,2)=ASN(NSTP-1)-ASN(NSTP)-OC(NSTP,1)
      WRITE(11=(NSTP+((NOC-1)*NTESTS)+(KTEST-1)*NTESTS*10)) OC(NSTP,1),
     2 OC(NSTP,2),ASN(NSTP),NSTP,ALAM,KTEST
399   IF( NTRYS .GT. 1 .AND. PTIMI .LE. 0.0)
     1  READ(11=(NSTP+((NOC-1)*NTESTS)+(KTEST-1)*NTESTS*10))
     2  OC(NSTP,1),OC(NSTP,2),ASN(NSTP),NCARE,GLAM,LOSCAS
      WRITE(7,208) NSTP,OC(NSTP,1),OC(NSTP,2),ASN(NSTP)
208   FORMAT(5X,I5,5X,E20.10,5X,E20.10,5X,E20.10)
                                                                    S
400   CONTINUE
C

C         THIS PART OF THE PROGRAM CALCULATES
C         E(N)ALAM)=AVERAGE SAMPLE NUMBER WHEN LAMDA=ALAM
C                                  AND
C         M(N)ALAM)=MEDIAN SAMPLE NUMBER WHEN LAMDA=ALAM
C                    AS WELL AS
C         OC(ALAM)=PROB(REJECTING H0)LAMDA=ALAM)
C         B(ALAM)=1-OC(ALAM)
C
C
      OCF=0.0
      AVR=1.0
      POW=0.0
      TMED=0.0
      DO 490 IN=1,MTP
      AVR=AVR+ASN(IN)
      OCF=OCF+OC(IN,1)
      POW=POW+OC(IN,2)
      TES=IN-1.0-AVR
      IF(TES .LT. 0.5 .OR. ( TMED .GT. 0.0 .AND. TES .GT. 0.5))GO TO 490
      TMED=IN
      IF(TES .GT. 0.5) TMED=IN-0.5
490   CONTINUE
      WRITE(IRITE,209) ALAM,TMED,AVR,OCF,POW
209   FORMAT(5X,F6.4,9X,F6.2,8X,F6.4,8X,F6.4,8X,F6.4)
500   CONTINUE
9112  CONTINUE
9113  CONTINUE
      WRITE(8,9117) IRP1,IRP2
9117  FORMAT(1H1,20X,"MISTAKES IN THEORY",I8,5X,I8)
      STOP
      END
C
C
C         THE SUBROUTINES CALLED FOLLOW
```

```
                                                                      START OF SEGMENT
      SUBROUTINE FSTEPPROB(PACC,PREJ,PCON,N)                          000369
C                                                                     000370
C        THIS SUBROUTINE CALCULATES                                   000371
C        THE PROBABILITIES OF ACCEPTING,REJECTING,                    000372
C        AND CONTINUING FOR STEPS                                     000373
C        FOR STEPS LESS THAN AND EQUAL TO THE FIRST                   000374
C        STEP AT WHICH A DECISION CAN BE MADE                         000375
C        THIS IS ACCOMPLISHED BY MEANS OF AN INFINITE                 000376
C        SUM OF INCOMPLETE BETA FUNCTIONS                             000377
C                                                                     000378
      COMMON /CB4/DEGF,ALAM                                           000379
      COMMON /CB5/ REG(30,2)                                          000380
      DO 50 IB=1,2                                                    000381
      TSUM=0.0                                                        000382
      IF( IB .EQ. 1 .AND. REG(N,1) .LE. 0.0) GO TO 30                 000383
      IF( IB .EQ. 2 .AND. REG(N,2) .GE. 1.E6) GO TO 30                000384
      FO=((DEGF*(FLOAT(N)-1.))/(DEGF-1.))*REG(N,IB)                   000385
      UO=1.0/(1.0+(((DEGF-1.0)/(DEGF*(FLOAT(N)   -1.)))*FO))         000386
         W1=(DEGF*(FLOAT(N)-1.))*0.5                                  000387
      W2=(DEGF-1.)*0.5                                                000388
      TSUM=BETINC(0,W1,W2,UO)                                         000389
      IF( ALAM .LE. 0.0) GO TO 20                                     000390
      DO 10 JR=1,101                                                  000391
      W2=((DEGF-1.)*0.5)+FLOAT(JR)                                    000392
      TOT=(FLOAT(JR)*ALOG(0.5*FLOAT(   N )*ALAM))+ALOG(BETINC(0,W1,W2,UO)000393
     1 ) -ALGAMA(FLOAT(JR+1))                                         000394
      TOT=EXP(TOT)                                                    000395
      TSUM=TSUM+TOT                                                   000396
      IF( TOT .LE. 1.E-06) GO TO 20                                   000397
10    CONTINUE                                                        000398
20    TSUM=TSUM*EXP(-.5*FLOAT(N)*ALAM)                                000399
30    IF( IB .GT. 1) GO TO 40                                         000400
      PACC=1.-TSUM                                                    000401
      GO TO 50                                                        000402
40    PREJ=TSUM                                                       000403
50    CONTINUE                                                        000404
51    PCON=1.-PACC-PREJ                                               000405
      RETURN                                                          000406
      END                                                             000407
                                                                      SEGMENT
```

```
                                                                    START OF SEGMENT
      FUNCTION BETINC(IND,A,B,X)                                    00040000
C     INCOMPLETE BETA FUNCTION AND ITS INVERSE                      00040900
C     MARK=1 FOR INVERSE (SEND DOWN PROB)                           00041000
C                                                                   00041100
C        THIS SUBFUNCTION CALCULATES THE INCOMPLETE BETA FUNCTION   00041200
C        THIS IS NEEDED TO CALCULATE THE PA,PR,PC AT THE FIRST STEP 00041300
C        A DECISION CAN BE MADE                                     00041400
C                                                                   00041500
      CAB=CGAM(A+B)-CGAM(A)-CGAM(B)-.5*ALOG((A+B)*6.28318531)       00041600
      IF(IND)10,10,20                                               00041700
10    EP=CAB+A*ALOG(X*(1.+B/A))+B*ALOG((1.-X)*(1.+A/B))             00041800
      IF(X-A/(A+B))12,12,14                                         00041900
12    BETINC=ZI(X,A,B)*EXP(EP+.5*ALOG(B/A))                         00042000
      RETURN                                                        00042100
14    BETINC=1.-ZI(1.-X,B,A)*EXP(EP+.5*ALOG(A/B))                   00042200
      RETURN                                                        00042300
20    IF(X-.5)22,22,24                                              00042400
22    QZ=ALOG(X)                                                    00042500
      IGO=1                                                         00042600
      AA=A                                                          00042700
      BB=B                                                          00042800
      GO TO 26                                                      00042900
24    QZ=ALOG(1.-X)                                                 00043000
      IGO=2                                                         00043100
      AA=B                                                          00043200
      BB=A                                                          00043300
26    XT=AA/(AA+BB)                                                 00043400
      CABB=CAB+.5*ALOG(BB/AA)+AA*ALOG(1.+BB/AA)+BB*ALOG(1.+AA/BB)   00043500
      DO 40 NC=1,100                                                00043600
      ZZ=ZI(XT,AA,BB)                                               00043700
      QX=CABB+AA*ALOG(XT)+BB*ALOG(1.-XT)+ALOG(ZZ)                   00043800
      XC=(QZ-QX)*(1.-XT)*ZZ/AA                                      00043900
      XC=AMAX1(XC,-.99)                                             00044000
      XC=AMIN1(XC,.5/XT-.5)                                         00044100
      XT=XT*(1.+XC)                                                 00044200
      IF(ABS(XC)-1.E-6)42,40,40                                     00044300
40    CONTINUE                                                      00044400
42    GO TO (44,46),IGO                                             00044500
44    BETINC=XT                                                     00044600
      RETURN                                                        00044700
46    BETINC=1.-XT                                                  00044800
      RETURN                                                        00044900
      END                                                           00045000
                                                                    SEGMENT
```

```
      SUBROUTINE BOOK (L1,L2,L3,LN,WN,QN,RN)                            START OF SEGM
C                                                                       00048
C                                                                       00048
C        THIS SUBROUTINE IS A BOOK KEEPING ROUTINE                      00048
C                                                                       00048
C        THIS ROUTINE CONVERTS A POINT IN THE GRID F(W,Q,R) TO          00049
C        A POINT IN THE RANDOM ACCESS DISK FILE                         00049
C        THE POINT IN THAT FILE FOR A PARTICULAR POINT                  00049
C        IS TERMED JREC                                                 00049
      COMMON /CB2/JREC,TSTAT                                            00049
      COMMON /CB1/GRIDW,GRIDQ,GRIDR                                     00049
      COMMON /CB3/ NCONW,NCONQ,NSTRT                                    00049
      COMMON /CB12/NUMW,NUMQ,NUMR                                       00049
      WN=(L1-1+NCONW)*GRIDW                                             00049
      QN=(L2-1+NCONQ)*GRIDQ                                             00049
      ENTRY RCAL(L1,L2,L3,LN,WN,QN,RN)                                  00049
      RN=(IFIX((WN**2.+QN**2.)/(NSTRT*GRIDR))+L3)*GRIDR                 00050
      IF(  LN .LE. 0)  GO TO 10                                         00050
      ENTRY CRITV( WN,QN,RN,LN)                                         00050
      DBLCHK=((LN*RN)-(WN**2.)-(QN**2.))*2.0                            00050
      IF( DBLCHK  .LE. 0.0)  GO TO 5                                    00050
      TSTAT=((WN-QN)**2.0)/DBLCHK                                       00050
      RETURN                                                            00050
    5 TSTAT= 1.0 E20                                                    00050
      RETURN                                                            00050
      ENTRY IENT(L1,L2,L3,WN,QN,RN)                                     00050
      L1=(WN/GRIDW)+1-NCONW                                             00051
      L2=(QN/GRIDQ)+1-NCONQ                                             00051
      L3=(RN/GRIDR)-IFIX((WN**2.+QN**2.)/(NSTRT*GRIDR))                 00051
      SZCHK=L1+((L2-1)*NUMW)+((L3-1)*NUMW*NUMQ)                         00051
      IF(ABS(SZCHK) .LT. 549755813886) GO TO 30                         00051
      JREC=541111111111                                                 00051
      RETURN                                                            00051
   30 JREC= IFIX(SZCHK)                                                 00051
      RETURN                                                            00051
   10 JREC=L1+((L2-1)*NUMW)+((L3-1)*NUMW*NUMQ)                          00051
      RETURN                                                            00052
      END                                                               00052
                                                                        00052
                                                                        00052
                                                                        00052
                                                                        SEGMENT
```

```
                                                                  START OF SEGMENT
      FUNCTION POSPROB(NV,QV,RV,N)                                00034400
C                                                                 00034500
C        THIS IS A FUNCTION TO DETERMINE IF                       00034600
C        A POINT IS ALLOWABLE AT STEP N                           00034700
C                                                                 00034800
      POSPROB=RV-((NV**2.+QV**2.)/FLOAT(N))                       00034900
      IF( ABS(POSPROB) .LE. 1.E-4) POSPROB=0.0                    00035000
      RETURN                                                      00035100
      END                                                         00035200
                                                                  SEGMENT
```

```
                                                                  START OF SEGMENT
      FUNCTION PHI(Y,XBAR,SIG)                                    00035300
C                                                                 00035400
C        THIS SUBFUNCTION CALCULATES THE NORMAL DENSITY FUNCTION  00035500
C                                                                 00035600
      PHI=0.3989422*EXP(-.5*(((Y-XBAR)/SIG)**2.))*(1./SIG)        00035700
      RETURN                                                      00035800
      END                                                         00035900
                                                                  SEGMENT
```

```
                                                                  START OF SEGMENT
      FUNCTION CHISQ(Y,DOF)                                       00036000
C                                                                 00036100
C        THIS SUBFUNCTION CALCULATES THE CHISQUARE DENSITY FUNCTION 00036200
C                                                                 00036300
      CHISQ          =((Y**((DOF/2.)-1.))*EXP(-Y/2.))/((2.**(DOF/2.)) 00036400
     1               *GAMMA(DOF/2.0))                            00036500
      IF( Y .EQ. 0.0 .AND. DOF .EQ. 2.0) CHISQ=0.5                00036600
      RETURN                                                      00036700
      END                                                         00036800
                                                                  SEGMENT
```

```
      FUNCTION DENSTS(A,B,C,KREC)                                       START OF SEGMENT
C                                                                       00052400
C                                                                       00052500
C        THIS SUBROUTINE CALCULATES FN(A,B,C)                           00052600
C        FROM G(A-Z,B-U,R-Z**2-U**2)*P(U)*P(Z)                          00052700
C        BY INTEGRATING OVER THE APPROPRIATE REGIONS                    00052800
      COMMON /CB1/ GRIDW,GRIDQ,GRIDR                                    00052900
      COMMON /CB6/ TMAXW,TMINW,TMAXQ,TMINQ                              00053000
      COMMON /CB7/ LSTP,ISUR                                            00053100
      COMMON /CB8/ JOINT,ICAL                                           00053200
      COMMON /CB9/ NIP,RINT,CUR                                         00053300
      COMMON /CB5/ REG(30,2)                                            00053400
      DIMENSION POINT(4,4),FVAL(4),RINT(5,4)                            00053500
      VOLUME=0.0                                                        00053600
C                                                                       00053700
C        THIS NUMERICAL INTEGRATION INVOLVES SUMMING THE VOLUMES        00053800
C        OF    TRAPEZOIDSC                                             00053900
C        U=ODIMENSION  = MEAN Z                                         00054000
C        Z=  W DIMENSION = MEAN 1                                       00054100
C                                                                       00054200
      CALL UBOUND(A,B,C,REG(LSTP,1),REG(LSTP,2),NREG,RINT)              00054300
      IF( NREG .EQ. 0)  GO TO 230                                       00054400
      DO 229 NIP=1,NREG                                                 00054500
      CALL ZRANGE(RINT(NIP,1),RINT(NIP,3),RINT(NIP,4),TMX,TMIN)         00054600
      USTRT= (IFIX(RINT(NIP,1)/GRIDW)+1)*GRIDW                          00054700
      IF(RINT(NIP,1) .LT. 0.0 .AND. RINT(NIP,1) .NE. (USTRT-GRIDW))     00054800
     1         USTRT=IFIX(RINT(NIP,1)/GRIDW)*GRIDW                      00054900
      UFIN=IFIX(RINT(NIP,2)/GRIDW)*GRIDW                                00055000
      IF(RINT(NIP,2) .LT. 0.0) UFIN=(IFIX(RINT(NIP,2)/GRIDW)-1)*GRIDW   00055100
      IF( UFIN .EQ. RINT(NIP,2)) UFIN=UFIN-GRIDW                        00055200
      IF( UFIN .LE. USTRT ) GO TO 220                                   00055300
      IF( LSTP .EQ. ISUR) GO TO 157                                     00055400
      IF((B-UFIN) .GT. TMAXQ .AND. (B-USTRT) .GT. TMAXQ) GO TO 220      00055500
      IF((B-UFIN) .LT. TMINQ .AND. (B-USTRT) .LT. TMINQ) GO TO 220      00055600
      IF((B-USTRT) .LT. TMINQ) USTRT=B-TMINQ                            00055700
      IF((B-USTRT) .GT. TMAXQ) USTRT=B-TMAXQ                            00055800
      IF((B-UFIN) .LT. TMINQ) UFIN=B-TMINQ                              00055900
      IF((B-UFIN) .GT. TMAXQ) UFIN=B-TMAXQ                              00056000
      IF( USTRT .GE. UFIN ) GO TO 220                                   00056100
157   U1=USTRT                                                          00056200
      U2=USTRT+GRIDW                                                    00056300
      CALL   ZRANGE(U1,RINT(NIP,3),RINT(NIP,4),ZMAX1,ZMIN1)             00056400
      POINT(1,1)=RINT(NIP,1)                                            00056500
      POINT(1,2)=TMX                                                    00056600
      POINT(2,1)=U1                                                     00056700
      POINT(2,2)=ZMAX1                                                  00056800
      POINT(3,1)=RINT(NIP,1)                                            00056900
      POINT(3,2)=TMIN                                                   00057000
      IF( TMX .EQ. TMIN) GO TO 158                                      00057100
      POINT(4,1)=U1                                                     00057200
      POINT(4,2)=ZMIN1                                                  00057300
      CUR=RINT(NIP,3)                                                   00057400
      CALL RESVOL(A,B,C,VOLUME,4,4,POINT,FVAL)                          00057500
      VOLUME=VOLUME+                                                    00057600
     2(AREA(RINT(NIP,4),A-POINT(4,2),B-POINT(4,1),RINT(NIP,4),A-POINT(3,00057700
     52),B-POINT(3,1))*(FVAL(3)+FVAL(4))*0.5)                           00057800
      NSIDES=4                                                          00057900
      GO TO 159                                                         00058000
158   POINT(3,1)=U1                                                     00058100
      POINT(3,2)=ZMIN1                                                  00058200
      CALL RESVOL(A,B,C,VOLUME,3,3,POINT,FVAL)                          00058300
      NSIDES=3                                                          00058400
159   ZBEG1= IFIX(ZMIN1/GRIDQ)*GRIDQ                                    00058500
      ZFIN1=IFIX( ZMAX1/GRIDQ)*GRIDQ                                    00058600
```

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160   CALL  ZRANGE(U2,RINT(NIP,3),RINT(NIP,4),ZMAX2,ZMIN2)                    00058900
      ZBEG2=IFIX(ZMIN2/GRIDQ)*GRIDQ                                           00059000
      ZFIN2=IFIX(ZMAX2/GRIDQ)*GRIDQ                                           00059100
      IF(ZMIN2 .GT. 0.0) ZBEG2=(IFIX(ZMIN2/GRIDW)+1)*GRIDW                    00059200
      IF(ZMAX2 .LT. 0.0) ZFIN2=(IFIX(ZMAX2/GRIDW)-1)*GRIDW                    00059300
      ZBEG=  AMAX1(ZBEG1,ZBEG2)                                               00059400
      ZFIN=  AMIN1(ZFIN1,ZFIN2)                                               00059500
      IF( ZBEG .GE. ZFIN) GO TO 197                                           00059600
      IF( LSTP .EQ. ISUR) GO TO 168                                           00059700
      IF((A-ZBEG) .GT. TMAXW .AND. (A-ZFIN) .GT. TMAXW) GO TO 197             00059800
      IF((A-ZBEG) .LT. TMINW .AND. (A-ZFIN) .LT. TMINW) GO TO 197             00059900
      IF((A-ZBEG) .LT. TMINW) ZBEG=A-TMAXW                                    00060000
      IF((A-ZBEG) .LT. TMINW) ZBEG=A-TMINW                                    00060100
      IF((A-ZFIN) .GT. TMAXW) ZFIN=A-TMAXW                                    00060200
      IF((A-ZFIN) .LT. TMINW) ZFIN=A-TMINW                                    00060300
      IF(ZBEG .GE. ZFIN) GO TO 197                                            00060400
168   ZINT=ZBEG                                                               00060500
170   Y1=TERPOS(U1,ZINT)                                                      00060600
      Y2=TERPOS(U2,ZINT)                                                      00060700
1178  IF(ZINT .NE. ZBEG .AND. ZINT .NE. ZFIN) GO TO 180                       00060800
      ZDET=ZMIN2                                                              00060900
      NPFI=1                                                                  00061000
      IF( ZINT .NE. ZBEG .OR. U1 .NE. USTRT) GO TO 174                        00061100
      UBFU=U1                                                                 00061200
      ZBFU=ZMAX1                                                              00061300
      FBFU=FVAL(2)                                                            00061400
      CUR=RINT(NIP,4)                                                         00061500
      IF( NSIDES .EQ. 3) GO TO 173                                            00061600
      POINT(2,1)=POINT(4,1)                                                   00061700
      POINT(2,2)=POINT(4,2)                                                   00061800
      FVAL(2)=FVAL(4)                                                         00061900
      GO TO 176                                                               00062000
173   POINT(2,1)=U1                                                           00062100
      POINT(2,2)=ZMIN1                                                        00062200
      FVAL(2)=FVAL(3)                                                         00062300
      IF( POINT(3,1) .NE. U1 .OR. POINT(3,2) .NE. ZMIN1)                      00062400
     1 FVAL(2)=TERPOS(U1,ZMIN1)                                               00062500
      IF( POINT(2,1) .NE. U1 .OR. POINT(2,2) .NE. ZMAX1)                      00062600
     2 FBFU=TERPOS(U1,ZMAX1)                                                  00062700
      GO TO 176                                                               00062800
174   IF( ZINT .NE. ZBEG) GO TO 175                                           00062900
      POINT(2,1)=UBFL                                                         00063000
      POINT(2,2)=ZBFL                                                         00063100
      FVAL(2)=FBFL                                                            00063200
      CUR=RINT(NIP,4)                                                         00063300
      GO TO 176                                                               00063400
175   POINT(2,1)=UBFU                                                         00063500
      POINT(2,2)=ZBFU                                                         00063600
      FVAL(2)=FBFU                                                            00063700
      ZDET=ZMAX2                                                              00063800
      CUR=RINT(NIP,3)                                                         00063900
176   POINT(1,1)=U2                                                           00064000
      POINT(1,2)=ZDET                                                         00064100
      POINT(3,1)=U2                                                           00064200
      POINT(3,2)=ZINT                                                         00064300
      FVAL(3)=Y2                                                              00064400
      POINT(4,1)=U1                                                           00064500
      POINT(4,2)=ZINT                                                         00064600
      FVAL(4)=Y1                                                              00064700
      NSIDES=4                                                                00064800
      CALL RESVOL(A,B,C,VOLUME,NSIDES,NPFI,POINT,FVAL)                        00064900
      IF( ZINT .NE. ZBEG) GO TO 177                                           00065000
      UBFL=POINT(I,1)                                                         00065100
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      GO TO 178                                                          00065400
  177 UBFU=POINT(1,1)                                                    00065500
      ZBFU=POINT(1,2)                                                    00065600
      FBFU=FVAL(1)                                                       00065700
  178 VOLUME=VOLUME+GRIDW*GRIDQ*1.5*(1.0/6.0)*(Y1+Y2)                    00065800
        IF(ZINT .EQ. ZFIN)  GO TO 200                                    00065900
      GO TO 190                                                          00066000
  180 VOLUME=VOLUME+(1.0/6.0)*GRIDQ*GRIDW*3.0*(Y1+Y2)                    00066100
  190 ZINT=ZINT+GRIDQ                                                    00066200
      IF(ZINT .LE. ZFIN) GO TO 170                                       00066300
      GO TO 200                                                          00066400
  197 IF( (ZMAX2 .EQ. ZMIN2) .AND. U2 .EQ. UFIN) GO TO 201               00066500
      POINT(1,1)=U1                                                      00066600
      POINT(1,2)=ZMAX1                                                   00066700
      POINT(2,1)=U2                                                      00066800
      POINT(2,2)=ZMAX2                                                   00066900
      POINT(3,1)=U1                                                      00067000
      POINT(3,2)=ZMIN1                                                   00067100
      POINT(4,1)=U2                                                      00067200
      POINT(4,2)=ZMIN2                                                   00067300
      COR=RINT(NIP,3)                                                    00067400
      CALL RESVOL(A,B,C,VOLUME,4,4,POINT,FVAL)                           00067500
      VOLUME=VOLUME+                                                     00067600
     2(AREA(RINT(NIP,4),A,POINT(4,2),B,POINT(4,1),RINT(NIP,4),A,POINT(3,00067700
     52),B,POINT(3,1))*(FVAL(3)+FVAL(4))*0.5)                            00067800
      UBFU=U2                                                            00067900
      ZBFU=ZMAX2                                                         00068000
      FBFU=FVAL(2)                                                       00068100
      UBFL=U2                                                            00068200
      ZBFL=ZMIN2                                                         00068300
      FBFL=FVAL(4)                                                       00068400
  200 U1=U2                                                              00068500
      U2=U2+GRIDW                                                        00068600
      ZBEG1=ZBEG2                                                        00068700
      ZFIN1=ZFIN2                                                        00068800
      ZMIN1=ZMIN2                                                        00068900
      ZMAX1=ZMAX2                                                        00069000
      IF( U2  .LE.  UFIN) GO TO 160                                      00069100
  201 CALL ZRANGE(RINT(NIP,2),RINT(NIP,3),RINT(NIP,4),TMX,TMIN)          00069200
      POINT(1,1)=RINT(NIP,2)                                             00069300
      POINT(1,2)=TMX                                                     00069400
      POINT(2,1)=U1                                                      00069500
      POINT(2,2)=ZMAX1                                                   00069600
      POINT(3,1)=RINT(NIP,2)                                             00069700
      POINT(3,2)=TMIN                                                    00069800
      IF(TMX .LE. TMIN) GO TO 219                                        00069900
      POINT(4,1)=U1                                                      00070000
      POINT(4,2)=ZMIN1                                                   00070100
      COR=RINT(NIP,3)                                                    00070200
      CALL RESVOL(A,B,C,VOLUME,4,4,POINT,FVAL)                           00070300
      VOLUME=VOLUME+                                                     00070400
     2(AREA(RINT(NIP,4),A,POINT(4,2),B,POINT(4,1),RINT(NIP,4),A,POINT(3,00070500
     52),B,POINT(3,1))*(FVAL(3)+FVAL(4))*0.5)                            00070600
      GO TO 229                                                          00070700
  219 POINT(3,1)=U1                                                      00070800
      POINT(3,2)=ZMIN1                                                   00070900
      CALL RESVOL(A,B,C,VOLUME,3,3,POINT,FVAL)                           00071000
      GO TO 229                                                          00071100
  220 UINT=(RINT(NIP,2)-RINT(NIP,1))*.5+RINT(NIP,1)                      00071200
      NTC=1                                                              00071300
      CALL ZRANGE(UINT,RINT(NIP,3),RINT(NIP,4),ZX,ZM)                    00071400
  221 IF( TMX .EQ. TMIN) GO TO 222                                       00071500
      NSIDES=4                                                           00071600
      POINT(1,1)=RINT(NIP+NTC)                                           00071700
                                                                         00071800
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```
      POINT(2,1)=UINT                                                    0007190
      POINT(2,2)=ZX                                                      0007200
      POINT(3,1)=RINT(NIP,NTC)                                           0007210
      POINT(3,2)=TMIN                                                    0007220
      POINT(4,1)=UINT                                                    0007230
      POINT(4,2)=ZM                                                      0007240
      CUR=RINT(NIP,3)                                                    0007250
      CALL RESVOL(A,B,C,VOLUME,NSIDES,4,POINT,FVAL)                      0007260
      VOLUME=VOLUME+                                                     0007270
     2(AREA(RINT(NIP,4),A-POINT(4,2),B-POINT(4,1),RINT(NIP,4),A-POINT(3,0007280
     52),B-POINT(3,1))*(FVAL(3)+FVAL(4))*0.5)                            0007290
      GO TO 223                                                          0007300
  222 NSIDES=3                                                           0007310
      POINT(1,1)=RINT(NIP,NTC)                                           0007320
      POINT(1,2)=TMX                                                     0007330
      POINT(2,1)=UINT                                                    0007340
      POINT(2,2)=ZX                                                      0007350
      POINT(3,1)=UINT                                                    0007360
      POINT(3,2)=ZM                                                      0007370
      CALL RESVOL(A,B,C,VOLUME,NSIDES,3,POINT,FVAL)                      0007380
  223 IF(NTC .GE. 2) GO TO 229                                           0007390
      CALL ZRANGE(RINT(NIP,2),RINT(NIP,3),RINT(NIP,4),TMX,TMIN)          0007400
      NTC=NTC+1                                                          0007410
      GO TO 221                                                          0007420
  229 CONTINUE                                                           0007430
  230 DENSTS=VOLUME                                                      0007440
      WRITE(ICAL=KREC) VOLUME                                            0007450
      RETURN                                                             0007460
      END                                                                0007470
                                                                        SEGMENT

                                                                  START OF SEGMENT
      FUNCTION CGAM(A)                                                   00046700
C                                                                        00046800
C     THIS SUBROUTINE IS NEEDED FOR THE INCOMPLETE BETA FUNCTION CALC    00046900
C                                                                        00047000
      AA=A                                                               00047100
      CAC=0.0                                                            00047200
      IF(A-2.)2,8,8                                                      00047300
    2 IF(A-1.)4,6,6                                                      00047400
    4 CAC=-2.*(A+.5)*ALOG(1.+1./A)+(A+1.5)*ALOG(1.+1./(A+1.))            00047500
      AA=A+2.                                                            00047600
      GO TO 8                                                            00047700
    6 CAC=-1.*(A+.5)*ALOG(1.+1./A)                                       00047800
      AA=A+1.                                                            00047900
    8 CA=2.269489/AA                                                     00048000
      CA=.52560647/(AA+1.0115231/(AA+1.5174737/(AA+CA)))                 00048100
      CA=.083333333/(AA+.03333333/(AA+.25238095/(AA+CA)))                00048200
      CGAM=CA+CAC                                                        00048300
      RETURN                                                             00048400
      END                                                                00048500
                                                                        SEGMENT
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      SUBROUTINE   UBOUND( A,B,C,VAN,VRN,NREG,RINT)                     START OF SEGMENT
      COMMON /CB7/ NSTP                                                 00074800
      COMMON /CB10/ CA,PA,CR,PR                                         00074900
      COMMON /CB11/ SINE45                                              00075000
      DIMENSION  RINT(5,5)                                              00075100
C        THIS SUBROUTINE CALCULATES THE INTEGRATION                     00075200
C        LIMITS OF U, AND ALSO DETERMINES THE                           00075300
C        NUMBER OF INTEGRATION REGIONS AND TYPE                         00075400
C        SO AS TO ALLOW DETERMINATION OF THE Z RANGE                    00075500
C        THESE ARE NEEDED TO OBTAIN THE DENSITY F(W,0,R) AT STEP        00075600
C        N FROM THE DENSITY AT STEP N-1                                 00075700
C        THE FOLLOWING CODE IS EMPLOYED                                 00075800
C        1=REJECTION ELLIPSE LOWER                                      00075900
C        2=REJECTION ELLIPSE UPPER                                      00076000
C        3=ACCEPTANCE ELLIPSE LOWER                                     00076100
C        4=ACCEPTANCE ELLIPSE UPPER                                     00076200
C        5=CIRCLE LOWER                                                 00076300
C        6=CIRCLE UPPER                                                 00076400
C        RINT(1,3)= UPPER Z CURVE ( LOWER W)                            00076500
C        RINT(2,4)= LOWER Z CURVE   ( UPPER W)                          00076600
      TERM2(XCR,XA,XB,XPR)= (XB*(XCR+1.)+XA*XPR)/(((XCR+1.0)**2.0)-(XPR 00076700
     1 **2.0))                                                          00076800
      DISCR(XCR,XA,XB,XPR,XC)= SQRT((TERM2(XCR,XA,XB,XPR)**2.0)-(((XA** 00076900
     1  2.0)-((XCR+1.0)*(XA**2.0+XB**2.0-XC)))/((XPR**2.)-((XCR+1.)**2.)00077000
     2 )))                                                              00077100
      IF( VAN .LE. 0.0)  GO TO 5                                        00077200
      DA=C-A**2.-B**2.+((.5*((A+B)**2.)*NSTP)/(NSTP+1.))+((.5*((A-B)**2.00077300
     5  ))/(CA+PA+1.))                                                 00077400
    5 IF( VRN .LE. 0.0)  GO TO 7                                        00077500
      DR=C-A**2.-B**2.+((.5*((A+B)**2.)*NSTP)/(NSTP+1.))                00077600
     7     +((.5*((A-B)**2.0))/(CR+PR+1.))                            00077700
    7 IF(VAN .LE. 0.0)  GO TO 60                                        00077800
      IF(VRN  .LE. 0.0)  GO TO 70                                       00077900
      IF(  DR .LE. 0.0 .AND. DA .LE. 0.0)  GO TO 15                     00078000
      IF( DA .LE. 0.0 .AND. DR .GT. 0.0) GO TO 60                       00078100
      DR1=SQRT((DR*NSTP)/(NSTP+1.0))                                    00078200
      DR2=SQRT(DR/(CR+PR+1.0))                                          00078300
      DA1=  SQRT((DA*NSTP)/(NSTP+1.0))                                  00078400
      DA2=SQRT(DA/(CA+PA+1.0))                                          00078500
      HA=( SINE45  *(A+B)*NSTP)/(NSTP+1.0)                              00078600
      HR=(SINE45  *(A+B)*NSTP)/(NSTP+1.0)                               00078700
      TKA=(SINE45  *(A-B))/(CA+PA+1.0)                                  00078800
      TKR=(SINE45  *(A-B))/(CR+PR+1.0)                                  00078900
      CHK=.5*(((A+B)*NSTP)**2.0)-NSTP*(NSTP+1.)*(A**2.+B**2.-C)         00079000
      IF(CHK)20,10,10                                                   00079100
   10 TLOC=(((HR-HA)**2.)/(DR1**2.))+((TKR-TKA)**2.)/(DR2**2.) -1.      00079200
      IF( TLOC .GT. 0.0)  GO TO 60                                      00079300
      TLOC=(((HA-HR)**2.)/(DA1**2.))+((TKA-TKR)**2.)/(DA2**2.) -1.      00079400
      TSPEC= (((HA-HR)**2.)/(DA1**2.))+((TKR+DR2-TKA)**2.)/(DA2**2.) -1.00079500
      IF( TLOC .LT. 0.0 .AND. TSPEC .GT. 0.0) GO TO 20                  00079600
   15 NREG=0                                                            00079700
      RETURN                                                            00079800
   20 CON=TERM2(CR,A,B,PR)                                              00079900
      QUAD=DISCR(CR,A,B,PR,C)                                           00080000
      QNLR=CON-QUAD                                                     00080100
      QNUR=CON+QUAD                                                     00080200
      QUAD=DISCR(CA,A,B,PA,C)                                           00080300
      CON=TERM2(CA,A,B,PA)                                              00080400
      QNLA=CON-QUAD                                                     00080500
      QNUA=CON+QUAD                                                     00080600
      IF(CHK) 30,30,40                                                  00080700
   30 NREG=4                                                            00080800
      RINT(1,1)=B-QNLA                                                  00080900
                                                                        00081000
                                                                        00081100
```

```
      RINT(1,3)=2.                                                    00081200
      RINT(1,4)=1.                                                    00081300
      RINT(2,1)=B-QNUA                                                00081400
      RINT(2,2)=B-QNLA                                                00081500
      RINT(2,3)=4.                                                    00081600
      RINT(2,4)=2.                                                    00081700
      RINT(3,1)=RINT(2,1)                                             00081800
      RINT(3,2)=RINT(2,2)                                             00081900
      RINT(3,3)=3.                                                    00082000
      RINT(3,4)=1.                                                    00082100
      RINT(4,1)=B-QNUR                                                00082200
      RINT(4,2)=B-QNUA                                                00082300
      RINT(4,3)=2.                                                    00082400
      RINT(4,4)=1.                                                    00082500
      RETURN                                                          00082600
40    NREG=5                                                          00082700
      RINT(1,1)=B-QNLA                                                00082800
      RINT(1,2)=B-QNLR                                                00082900
      RINT(1,3)=2.                                                    00083000
      RINT(1,4)=1.                                                    00083100
      RINT(2,1)=B-QNUA                                                00083200
      RINT(2,2)=B-QNLA                                                00083300
      RINT(2,3)=2.                                                    00083400
      RINT(2,4)=4.                                                    00083500
      RINT(5,1)=B-QNUR                                                00083600
      RINT(5,2)=B-QNUA                                                00083700
      RINT(5,3)=2.                                                    00083800
      RINT(5,4)=1.                                                    00083900
      CON= SINE45  *(A+B)*NSTP                                        00084000
      QUAD=SQRT(CHK)                                                  00084100
      QN1=(1./(NSTP+1.))*(CON+QUAD)* SINE45                           00084200
      QN2=(1./(NSTP+1.))*(CON-QUAD)*SINE45                            00084300
      RINT(3,1)=B-QN1                                                 00084400
      RINT(3,2)=B-QNLA                                                00084500
      RINT(4,1)=B-QNUA                                                00084600
      RINT(4,2)=B-QN2                                                 00084700
      IF(IKR  .LT. IKA)  GO TO 50                                     00084800
      RINT(3,3)=3.                                                    00084900
      RINT(3,4)=1.                                                    00085000
      RINT(4,3)=3.                                                    00085100
      RINT(4,4)=1.                                                    00085200
      RETURN                                                          00085300
50    RINT(3,3)=1.                                                    00085400
      RINT(3,4)=3.                                                    00085500
      RINT(4,3)=1.                                                    00085600
      RINT(4,4)=3.                                                    00085700
      RETURN                                                          00085800
60    IF(  DR  .LE. 0.0)  GO TO 15                                    00085900
      NREG=1                                                          00086000
      CON=TERM2(CR,A,B,PR)                                            00086100
      QUAD=DISCR(CR,A,B,PR,C)                                         00086200
      QNLR=CON-QUAD                                                   00086300
      QNUR=CON+QUAD                                                   00086400
      RINT(1,1)=B-QNUR                                                00086500
      RINT(1,2)=B-QNLR                                                00086600
      RINT(1,3)=2.                                                    00086700
      RINT(1,4)=1.                                                    00086800
      RETURN                                                          00086900
70    IF(DA  .GT. 0.0)  GO TO 80                                      00087000
      NREG=1                                                          00087100
      RINT(1,1)=- SQRT(C)                                             00087200
      RINT(1,2)=SQRT(C)                                               00087300
      RINT(1,3)=6.                                                    00087400
      RINT(1,4)=5                                                     00087500
```

```
80      NREG=4                                                          00087700
        CON=TERM2(CA,A,B,PA)                                            00087800
        QUAD=DISCR(CA,A,B,PA,C)                                         00087900
        QNLA=CON-QUAD                                                   00088000
        QNUA=CON+QUAD                                                   00088100
        RINT(1,1)=B-QNLA                                                00088200
        RINT(1,2)=SQRT(C)                                               00088300
        RINT(1,3)=6.                                                    00088400
        RINT(1,4)=5.                                                    00088500
        RINT(2,1)=B-QNUA                                                00088600
        RINT(2,2)=B-QNLA                                                00088700
        RINT(2,3)=6.                                                    00088800
        RINT(2,4)=4.                                                    00088900
        RINT(3,1)=B-QNUA                                                00089000
        RINT(3,2)=B-QNLA                                                00089100
        RINT(3,3)=3.                                                    00089200
        RINT(3,4)=5.                                                    00089300
        RINT(4,1)=-  SQRT(C)                                            00089400
        RINT(4,2)=B-QNUA                                                00089500
        RINT(4,3)=6.                                                    00089600
        RINT(4,4)=5.                                                    00089700
        RETURN                                                          00089800
        END                                                             00089900
                                                                      SEGMENT
```

```
                                                                  START OF SEGMENT
        FUNCTION ZI(X,A,B)                                              00045100
C                                                                       00045200
C       THIS SUBROUTINE IS NEEDED FOR THE INCOMPLETE BETA FUNCTION CALC 00045300
C                                                                       00045400
        FN=.7*(ALOG(15.+A+B))**2+AMAX1(X*(A+B)-A,0.0)                   00045500
        N=INT(FN)                                                       00045600
        C=1.-(A+B)*X/(A+2.*FN)                                          00045700
        ZI=2./(C+SQRT(C**2-4.*FN*(FN-B)*X/(A+2.*FN)**2))                00045800
        DO 60 J=1,N                                                     00045900
        FN=N+1-J                                                        00046000
        A2N=A+2.*FN                                                     00046100
        ZI=(A2N-2.)*(A2N-1.-FN*(FN-B)*X*ZI/A2N)                         00046200
        ZI=1./(1.-(A+FN-1.)*(A+FN-1.+B)*X/ZI)                           00046300
60      CONTINUE                                                        00046400
        RETURN                                                          00046500
        END                                                             00046600
                                                                      SEGMENT
```

```
      SUBROUTINE RESVOL(A,B,C,VOLUME,NSIDES,NPNTBT,POINT,FVAL)                  00093...
      COMMON /CB9/NIP,RINT,CUR                                                  00093400
      COMMON /CB1/ GRIDW,GRIDQ,GRIDR                                            00093500
      DIMENSION POINT(4,4),FVAL(4),RINT(5,4) ,S(4)                              00093600
      TLINE(X1,Y1,X2,Y2,X)=((Y2-Y1)*(X-X1)/(X2-X1))+Y1                          00093700
                                                                                00093800
      DO 10 IP=1,NPNTBT                                                         00093900
      FVAL(IP)=TERPOS(POINT(IP,1),POINT(IP,2))                                  00094000
10    CONTINUE                                                                  00094100
      GO TO (40,40,20,30),NSIDES                                                00094200
20    PIEC1=AREA(RINT(NIP,3),A-POINT(1,2),B-POINT(1,1),RINT(NIP,3),A-           00094300
     1      POINT(2,2),B-POINT(2,1)       )                                    00094400
      PIEC2=AREA(RINT(NIP,4),A-POINT(1,2),B-POINT(1,1),RINT(NIP,4),A-           00094500
     1     POINT(3,2),B-POINT(3,1))                                             00094600
      VOLUME=VOLUME+(PIEC1*(FVAL(1)+FVAL(2))+PIEC2*(FVAL(1)+FVAL(3)))*.50094700
      CURQ=IFIX((POINT(2,2)-POINT(3,2))*0.5+POINT(3,2))/GRIDQ)*GRIDQ            00094800
      FIMP=TERPOS(POINT(2,1),CURQ)                                              00094900
      DO 21 I=1,2                                                               00095000
      DO 21 J=1,3                                                               00095100
      IF(FVAL(I) .GE. FVAL(J)) GO TO 21                                         00095200
      AINT1=FVAL(1)                                                             00095300
      FVAL(I)=FVAL(J)                                                           00095400
      FVAL(J)=AINT1                                                             00095500
      AINT1=POINT(I,1)                                                          00095600
      AINT2=POINT(I,2)                                                          00095700
      POINT(I,1)=POINT(J,1)                                                     00095800
      POINT(I,2)=POINT(J,2)                                                     00095900
      POINT(J,1)=AINT1                                                          00096000
      POINT(J,2)=AINT2                                                          00096100
21    CONTINUE                                                                  00096200
      S(1)=SQRT(((POINT(1,1)-POINT(2,1))**2.)+((POINT(1,2)-POINT(2,2))          00096300
     1    **2.0))                                                               00096400
      S(2)=SQRT(((POINT(1,1)-POINT(3,1))**2.)+((POINT(1,2)-POINT(3,2))          00096500
     1   **2.0))                                                                00096600
      S(3)=SQRT(((POINT(2,1)-POINT(3,1))**2.)+((POINT(2,2)-POINT(3,2))          00096700
     1   **2.0))                                                                00096800
      SPER=0.5*(S(1)+S(2)+S(3))                                                 00096900
      BSAREA=     (SPER*(SPER-S(1))*(SPER-S(2))*(SPER-S(3)))                    00097000
      IF( BSAREA .LE. 0.0) BSAREA=0.0                                           00097100
      BSAREA=SQRT(BSAREA)                                                       00097200
                                                                                00097300
      IF( FVAL(1) .LE. 0.0) GO TO 25                                            00097400
      RVOL=((FVAL(1)-FVAL(2)+FVAL(1)-FVAL(3))*BSAREA)/3.0                       00097500
      VOLUME=VOLUME+((BSAREA*FVAL(1))-RVOL)                                     00097600
25    EXTRA=(BSAREA+PIEC1+PIEC2)*(FIMP-AMIN1(FVAL(1),FVAL(2),FVAL(3)))          00097700
     1 *      0.5                                                               00097800
      EXTRA=AMAX1(0.0,EXTRA)                                                    00097900
      VOLUME=VOLUME+EXTRA                                                       00098000
40    RETURN                                                                    00098100
30    H=ABS(POINT(1,1)-POINT(2,1))                                              00098200
      IF( POINT(1,1) .EQ. POINT(2,1) .OR. POINT(3,1) .EQ. POINT(4,1))           00098300
     1                RETURN                                                   00098400
      B1=0.5*ABS(POINT(2,2)-POINT(4,2))*(FVAL(2)+FVAL(4))                       00098500
      BP=0.5*ABS(POINT(1,2)-POINT(3,2))*(FVAL(1)+FVAL(3))                       00098600
      X=0.5*(POINT(2,1)-POINT(1,1))+POINT(1,1)                                  00098700
      Y1=TLINE(POINT(1,1),POINT(1,2),POINT(2,1),POINT(2,2),X)                   00098800
      Z1=TLINE(POINT(1,1),FVAL(1),POINT(2,1),FVAL(2),X)                         00098900
      X=0.5*(POINT(4,1)-POINT(3,1))+POINT(3,1)                                  00099000
      Y2=TLINE(POINT(3,1),POINT(3,2),POINT(4,1),POINT(4,2),X)                   00099100
      Z2=TLINE(POINT(3,1),FVAL(3),POINT(4,1),FVAL(4),X)                         00099200
      BH=ABS(Y1-Y2)                                                             00099300
      BMID=0.5*BH*(Z1+Z2)                                                       00099400
```

```
      VOLUME=VOLUME+                                                    00099600
     1(AREA(CUR,A-POINT(2,2),B-POINT(2,1),CUR,A-POINT(1,2),B-POINT(1,1))00099700
     2        *(FVAL(1)+FVAL(2))*0.5)                                  00099800
      RETURN                                                            00099900
      END                                                               00100000
                                                                        SEGMENT
```

```
                                                                        START OF SEGMENT
      SUBROUTINE   ZRANGE(UVAL,TCL,TCU,ZMAX,ZMIN)                       00090000
      COMMON /CB10/CA,PA,CR,PR,A,B,C                                    00090100
C                                                                       00090200
C                                                                       00090300
C        THIS SUBROUTINE CALCULATES THE Z INTEGRATION LIMITS            00090400
C        FOR A GIVEN U VALUE                                            00090500
C                                                                       00090600
C                                                                       00090700
      ELLIPS(XPR,XCR,QP,POM)=((A+XPR*QP)/(XCR+1.))+((-1.)**POM)*        00090800
     1SQRT(ECHECK(1,(((A+XPR*QP)/(XCR+1.))**2.-(A**2.+B**2.-C-2.*B*QP   00090900
     2 +((XCR+1.)*(QP**2.)))/(XCR+1.))))                                00091000
      CIRCLE(QP,POM)=((-1.)**POM)*                                      00091100
     1 SQRT(ECHECK(2,(C-(QP**2.))))                                     00091200
      MTYP=IFIX(TCU)                                                    00091300
      LTYP=IFIX(TCL)                                                    00091400
      QNP=3-UVAL                                                        00091500
      GO TO (10,10,20,20,30,30) ,MTYP                                   00091600
10    ZCAL= ELLIPS(PR,CR,QNP,TCU)                                       00091700
      GO TO 40                                                          00091800
20    ZCAL=ELLIPS(PA,CA,QNP,TCU)                                        00091900
      GO TO 40                                                          00092000
30    ZCAL= CIRCLE(UVAL,TCU)                                            00092100
40    ZMAX=A-ZCAL                                                       00092200
      GO TO (50,50,60,60,70,70)  ,LTYP                                  00092300
50    ZCAL=ELLIPS(PR,CR,QNP,TCL)                                        00092400
      GO TO 80                                                          00092500
60    ZCAL=ELLIPS(PA,CA,QNP,TCL)                                        00092600
      GO TO 80                                                          00092700
70    ZCAL= CIRCLE(UVAL,TCL)                                            00092800
80    ZMIN=A-ZCAL                                                       00092900
      RETURN                                                            00093000
      END                                                               00093100
                                                                        SEGMENT
```

AD-A072 641 UNION COLL AND UNIV SCHENECTADY NY INST OF ADMINISTR--ETC F/G 12/1
AN EXACT TEST FOR THE SEQUENTIAL ANALYSIS OF VARIANCE.(U)
AUG 79 R W MILLER N00014-77-C-0438
UNCLASSIFIED AES-7906 NL

3 OF 3
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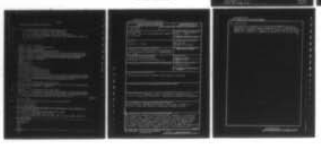

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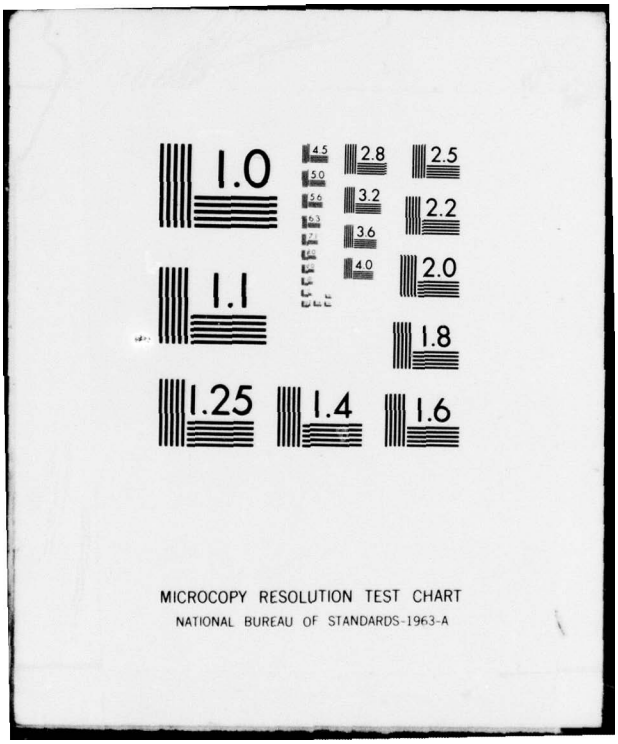
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5.0
5.6
MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

```
      FUNCTION TERPU(W,Q,R,IREAD)                                        00100100
C        THIS ROUTINE ESTIMATES THE DENSITY F(W,Q,R)                     00100200
C        FOR POINTS NOT LYING ON THE TRIVARIATE GRID                     00100300
C        THIS SUBPROGRAM PERFORMS INTERPOLATION IN                       00100400
C        ONE,TWO,OR THREE DIMENSIONS                                     00100500
      COMMON/CB1/GRIDW,GRIDQ,GRIDR                                       00100600
      COMMON /CB3/NCONW,NCONQ,NSTRT                                      00100700
      COMMON/CB14/XMEAN(2),XBR1,XBR2,VAR,DGF                             00100800
      COMMON/CB7/ LSTP,ISUR                                              00100900
      COMMON /CB6/ IMAXW,IMINW,IMAXQ,IMINQ                               00101000
      COMMON /CB12/ NUMW,NUMQ,NUMR                                       00101100
      COMMON /CB13/ QBEG,WBEG                                            00101200
      COMMON /CB2/ JREC,ISTAT                                            00101300
      COMMON /CB5/ REG(30,2)                                             00101400
      COMMON /CB20/ RECMAX
      DIMENSION COORD(8,4),XVAL(10),YVAL(10)                             00101500
      DET2(A,B,C,D)=A*B-C*D                                              00101600
      TERPU=0.0                                                          00101700
                                                                         00101800
      IF( POSPROB(W,Q,R,LSTP+1) .LT. 0.0) RETURN                         00101900
10    LB1=(W/GRIDW)+IFIX(NUMW/2.)+1-IFIX((ISUR*XMEAN(1))/GRIDW)          00102000
      WBEG=(LB1-1-IFIX(NUMW/2)+IFIX((ISUR*XMEAN(1))/GRIDW))*GRIDW        00102100
      LB2=(Q/GRIDQ)+IFIX(NUMQ/2)+1-IFIX((ISUR*XMEAN(2))/GRIDQ)           00102200
      QBEG=(LB2-1-IFIX(NUMQ/2)+IFIX((ISUR*XMEAN(2))/GRIDQ))*GRIDQ        00102300
      ENTRY TERPU1(W,Q,R,IREAD)                                          00102400
      JF=2                                                               00102500
      JS=1                                                               00102600
      J1=1                                                               00102700
      NPSF=1                                                             00102800
      NPNA=0                                                             00102900
      IF(WBEG .EQ. W .AND. QBEG .EQ. Q) GO TO 15                         00103000
      IF(WBEG .EQ. W) GO TO 45                                           00103100
      IF(QBEG .EQ. Q) GO TO 80                                           00103200
      GO TO 95                                                           00103300
15    IF((W .GT. TMAXW .OR. W .LT. TMINW)                                00103400
     1     .AND. (Q .GT. TMAXQ .OR. Q .LT. TMINQ)) GO TO 95              00103500
      IF(W .GT. TMAXW .OR. W .LT. TMINW) GO TO 80                        00103600
      IF( Q .GT. TMAXQ .OR. Q .LT. TMINQ) GO TO 45                       00103700
C                                                                        00103800
C        INTERPOLATION IN ONE DIMENSION(RN)                              00103900
C                  VIA 4TH DEGRE LAGRANGE                                00104000
C                                                                        00104100
      L4=(R/GRIDR)-IFIX(((W**2.)+(Q**2.))/(NSTRT*GRIDR))                 00104200
      L5=L4+2                                                            00104300
      POW=-1.0                                                           00104400
      IF(L5 .GT. NUMR) L5=NUMR                                           00104500
      IF(L4) 16,16,17                                                    00104600
16    L5=-1                                                              00104700
      POW=1.0                                                            00104800
17    WN=W                                                               00104900
      QN=Q                                                               00105000
      NP=1                                                               00105100
      NTT=NUMR/2                                                         00105200
      DENS=0.0                                                           00105300
18    DO 20 I=1,NTT                                                      00105400
      L6=L5+1+(POW*I)                                                    00105500
      IF(L6 .LE. 0 .OR. L6 .GT. NUMR) GO TO 19                           00105600
      CALL RCAL(0,0,L6,LSTP,WN,QN,RN)                                    00105700
      IF( ISTAT .GE. REG(LSTP,2)) GO TO 20                               00105800
      CALL IENT(LJ1,LJ2,LJ3,WN,QN,RN)                                    00105900
      READ(IREAD=JREC) YVAL(NP)                                          00106000
      XVAL(NP)=RN                                                        00106100
      IF(NP .GE. 4) GO TO 21                                             00106200
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 19     POW=(-1.0)*POW                                                          0010660
        GO TO 16                                                                0010670
 21     DENS=POLY(XVAL,YVAL,R,NP)                                               0010680
 40     IF( DENS .LT. 1.0E-35) DENS=0.0                                         0010690
        TERP0=DENS                                                              0010700
        RETURN                                                                  0010710
 45     IF(W .GT. TMAXW .OR. W .LT. TMINW) GO TO 95                             0010720
        V1=0                                                                    0010730
C                                                                               0010740
C          TWO DIMENSIONAL INTERPOLATION (QN,RN)                                0010750
C          USING A 4 POINT LATTICE FOR LAGRANGE                                 0010760
C          OR USING 3 POINT PLANAR IF ALL                                       0010770
C          POINTS ARE NOT AVAILABLE                                             0010780
C                                                                               0010790
        IF(Q .LT. 0.0) QBEG=QBEG-GRIDQ                                          0010800
        IF(QBEG .LT. TMINQ) QBEG=TMINQ+GRIDQ                                    0010810
        IF((QBEG+GRIDQ) .GT. TMAXQ) QBEG=TMAXQ-GRIDQ                            0010820
        RBEG=IFIX(R/GRIDR)*GRIDR                                                0010830
 47     DO 50 I1=1,JF,JS                                                        0010840
        QN=QBEG+(I1-J1)*GRIDQ                                                   0010850
        IF(QN .LT. TMINQ .OR. QN .GT. TMAXQ) GO TO 48                           0010860
        DO 50 I2=1,JF,JS                                                        0010870
        RN=RBEG+(I2-J1)*GRIDR                                                   0010880
        L4=(RN/GRIDR)-IFIX(((W**2.)+(QN**2.))/(NSTRT*GRIDR))                    0010890
        IF( L4 .LE. 0 .OR. L4 .GT. NUMR .OR. RN .LT. 0.0) GO TO 48              0010900
        CALL CRITV(WN,QN,RN,LSTP)                                               0010910
        IF(TSTAT .GE. REG(LSTP,2)) GO TO 48                                     0010920
        CALL IENT(JLW,JLQ,JLR,WN,QN,RN)                                         0010930
        READ(IRLAD=JRLC) COORD(NPSF,1)                                          0010940
        COORD(NPSF,2)=QN                                                        0010950
        COORD(NPSF,3)=RN                                                        0010960
        IF(NPSF .GE. 3 .AND. NPNA .GE. 1) GO TO 60                              0010970
        NPSF=NPSF+1                                                             0010980
        GO TO 50                                                                0010990
 48     NPNA=NPNA+1                                                             0011000
 50     CONTINUE                                                                0011010
        IF( NPSF .EQ. 4) GO TO 70                                               0011020
        IF( JF .GT. 1) GO TO 55                                                 0011030
        JF=4                                                                    0011040
        J1=2                                                                    0011050
        JS=2                                                                    0011060
        GO TO 47                                                                0011070
 55     RETURN                                                                  0011080
 60     PLANE=DET2((COORD(2,3)-COORD(1,3)),(COORD(3,1)-COORD(1,1)),(COORD(      0011090
       13,3)-COORD(1,3)),(COORD(2,1)-COORD(1,1)))*(V1-COORD(1,2))               0011100
        PLANE=PLANE+(DET2((COORD(2,1)-COORD(1,1)),(COORD(3,2)-COORD(1,2))       0011110
      1 ,(COORD(3,1)-COORD(1,1)),(COORD(2,2)-COORD(1,2))))*(R-COORD(3,1))       0011120
        PMULT=DET2((COORD(2,2)-COORD(1,2)),(COORD(3,3)-COORD(1,3)),(COORD(      0011130
      1 3,2)-COORD(1,2)),(COORD(3,2)-COORD(1,2)))                               0011140
        IF(PMULT .EQ. 0.0) GO TO 65                                             0011150
        DENS=COORD(1,1)-(PLANE/PMULT)                                           0011160
        GO TO 40                                                                0011170
 65     DENS=COORD(1,1)                                                         0011180
        GO TO 40                                                                0011190
 70     DO 75 I=2,4                                                             0011200
        IF( COORD(I,2) .NE. COORD(1,2)) X1=COORD(I,2)                           0011210
        IF(COORD(I,3) .NE. COORD(1,3)) Y1=COORD(I,3)                            0011220
        IF((COORD(I,2) .NE. COORD(1,2))                                         0011230
      1    .AND. (COORD(I,3) .EQ. COORD(1,3))) F1=COORD(I,1)                    0011240
        IF((COORD(I,2) .NE. COORD(1,2))                                         0011250
      1  .AND. (COORD(I,3) .NE. COORD(1,3))) F3=COORD(I,1)                      0011260
 75     CONTINUE                                                                0011270
        DENS=(1./((COORD(1,2)-X1)*(COORD(1,3)-Y1)))                             0011280
```

```
     1 *((V1-X1)*(R-Y1)*COORD(1,1)-((V1-COORD(1,2))*(R-Y1)*F1)-( (V1-X1)0011290
     2 *(R-COORD(1,3))*F2) +(V1-COORD(1,2))*(R-COORD(1,3))*F3)          0011300
       GO TO 40                                                        0011310
80     IF( Q .GT. TMAXQ .OR. Q .LT. TMINQ) GO TO 95                    0011320
       V1=W                                                            0011330
C                                                                      0011340
C         TWO DIMENSIONAL INTERPOLATION     WN,QN                    0011350
C                                                                      0011360
       IF( W .LT. 0.0) WBEG=WBEG-GRIDW                                 0011370
       IF( WBEG .LT. TMINW) WBEG=TMINW+GRIDW                           0011380
       IF((WBEG+GRIDW) .GT.TMAXW) WBEG=TMAXW-GRIDW                     0011390
       RBEG=IFIX(R/GRIDR)*GRIDR                                        0011400
83     DO 90 II=1,JF,JS                                                0011410
       WN=WBEG+(II-JI)*GRIDW                                           0011420
       IF( WN .LT. TMINW .OR. WN .GT. TMAXW) GO TO 85                  0011430
       DO 90 I2=1,JF,JS                                                0011440
       L4=(RN/GRIDR)-IFIX(((Q**2.)+(WN**2.))/(NSTRT*GRIDR))            0011450
       IF( L4 .LE. 0 .OR. L4 .GT. NUMR .OR. RN .LT. 0.0) GO TO 85      0011460
       CALL CRITV(WN,QN,RN,LSTP)                                       0011470
       IF( TSTAT .GE. REG(LSTP,2)) GO TO 85                            0011480
       CALL IENT(JLW,JLQ,JLR,WN,QN,RN)                                 0011490
       READ(IREAD=JRLC) COORD(NPSF,1)                                  0011500
       COORD(NPSF,2)=WN                                                0011510
       COORD(NPSF,3)=RN                                                0011520
       IF(NPSF .GE. 3 .AND. NPNA .GE. 1) GO TO 60                      0011530
       NPSF=NPSF+1                                                     0011540
       GO TO 90                                                        0011550
85     NPNA=NPNA+1                                                     0011560
90     CONTINUE                                                        0011570
       IF( NPSF .EQ. 4) GO TO 70                                       0011580
       IF( JF .GT. 1) GO TO 55                                         0011590
       JF=4                                                            0011600
       JI=2                                                            0011610
       JS=2                                                            0011620
       GO TO 83                                                        0011630
C                                                                      0011640
C         THREE DIMENSIONAL INTERPOLATION                              0011650
C         VIA AN 8 POINT LATTICE FOR LAGRANGE                          0011660
C         OR HYPERPLANAR INTERPOLATION                                 0011670
C         IF POINTS AREN'T AVAILABLE                                   0011680
C                                                                      0011690
95     IF(W .LT. 0.0) WBEG=WBEG-GRIDW                                  0011700
       IF(Q .LT. 0.0) QBEG=QBEG-GRIDQ                                  0011710
       IF(WBEG .LT. TMINW) WBEG=TMINW                                  0011720
       IF(QBEG .LT. TMINQ) QBEG=TMINQ                                  0011730
       IF((WBEG+GRIDW) .GT. TMAXW) WBEG=TMAXW-GRIDW                    0011740
       IF((QBEG+GRIDQ) .GT. TMAXQ) QBEG=TMAXQ-GRIDQ                    0011750
       RBEG=IFIX(R/GRIDR)*GRIDR                                        0011760
100    DO 120 II=1,JF,JS                                               0011770
       WN=WBEG+(II-JI)*GRIDW                                           0011780
       IF( WN .LT. TMINW .OR. WN .GT. TMAXW) GO TO 110                 0011790
       DO 120 I2=1,JF,JS                                               0011800
       QN=QBEG+(I2-JI)*GRIDQ                                           0011810
       IF(QN .LT. TMINQ .OR. QN .GT. TMAXQ) GO TO 110                  0011820
       DO 120 I3=1,JF,JS                                               0011830
       RN=RBEG+(I3-JI)*GRIDR                                           0011840
       IF(RN .LE. 0.0) GO TO 110                                       0011850
       L4=(RN/GRIDR)-IFIX(((WN**2.)+(QN**2.))/(NSTRT*GRIDR))           0011860
       IF( L4 .LE. 0 .OR. L4 .GT. NUMR) GO TO 110                      0011870
       NPSF=NPSF+1                                                     0011880
       CALL IENT(JLW,JLQ,JLR,WN,QN,RN)                                 0011890
       READ(IREAD=JRLC) COORD(NPSF,1)                                  0011900
       COORD(NPSF,2)=WN                                                0011910
       COORD(NPSF,3)=QN                                                0011920
```

```
      GO TO 120                                                          00119500
C
  110 NPNA=NPNA+1                                                        00119600
  120 CONTINUE                                                           00119700
      IF( NPSF .EQ. 8) GO TO 130                                         00119800
      IF( JF .GT. 1) GO TO 55                                            00119900
C
      JF=4                                                               00120000
      JI=2                                                               00120100
      JS=2                                                               00120200
      GO TO 150                                                          00120300
  130 DENS=0.0                                                           00120400
      DO 190 I=1,8                                                       00120500
C
      DO 181 J1=1,8                                                      00120600
      IF( COORD(J1,2) .NE. COORD(I,2)) GO TO 182                         00120700
  181 CONTINUE                                                           00120800
  182 DO 183 J2=1,8                                                      00120900
      IF( COORD(J2,3) .NE. COORD(I,3)) GO TO 184                         00121000
  183 CONTINUE                                                           00121100
C
  184 DO 185 J3=1,8                                                      00121200
      IF( COORD(J3,4) .NE. COORD(I,3)) GO TO 186                         00121300
  185 CONTINUE                                                           00121400
  186 DENS=DENS+(((R-COORD(I,2))*(Q-COORD(I,3))*(R-COORD(I,4)))          00121500
     1 /((COORD(I,2)-COORD(J1,2))*(COORD(I,3)-COORD(J2,3))*(COORD(I,4)-  00121600
     2 COORD(J3,3))))*COORD(I,1)                                         00121700
C
  190 CONTINUE                                                           00121800
      GO TO 40                                                           00121900
  245 CONTINUE                                                           00122000
      RETURN                                                             00122100
      END                                                                00122200
                                                                        SEGMENT
```

```
      FUNCTION AREA(WHR1,CORW1,CORQ1,WHR2,CORW2,CORQ2)                  00122300
      COMMON /CB10/ CA,PA,CR,PR,A,B,C                                   00122400
      Q1=CORQ1                                                          00122500
      Q2=CORQ2                                                          00122600
      W1=CORW1                                                          00122700
      W2=CORW2                                                          00122800
      AREA=0.0                                                          00122900
      IF(           (WHR1 .EQ. WHR2)                                    00123000
     1                        .OR.                                     00123100
     2     ((AMOD(WHR1,2) .EQ. 0.0) .AND. (WHR2 .EQ. (WHR1+1.)))       00123200
     4                        .OR.                                     00123300
     5     ((AMOD(WHR1,2) .EQ. 1.) .AND.(WHR2 .EQ.(WHR1-1.))) ) GO TO 10 00123400
      RETURN                                                            00123500
   10 IGO=IFIX(WHR2)                                                    00123600
      GO TO (20,20,30,30,30,30) ,IGO                                    00123700
   20 XC=CR                                                             00123800
      XP=PR                                                             00123900
      GO TO 40                                                          00124000
   30 XC=CA                                                             00124100
      XP=PA                                                             00124200
   40 IF((W1*W2) .LT. 0.0) GO TO 75                                     00124300
      EPART1=((A*(Q2-Q1)+0.5*XP*(Q2**2.-Q1**2.))/(XC+1.))               00124400
   45 C1=(XP**2.)-((XC+1.)**2.)                                         00124500
      C2=2.*(A*XP+B*(XC+1.))                                            00124600
      C3=A**2.-(XC+1.)*(A**2.+B**2.-C)                                  00124700
      TCURV=ABS(0.5*(Q2-Q1)*(W1+W2))                                    00124800
      TRM1=C1*(Q2**2.)+C2*Q2+C3                                         00124900
      IF(TRM1 .LT. 0.0 .AND. ABS(TRM1) .LT. 1.E-04) TRM1=0.0            00125000
      TRM1=(2.*C1*Q2+C2)*SQRT(TRM1)                                     00125100
      TRM2=C1*(Q1**2.)+C2*Q1+C3                                         00125200
      IF( TRM2 .LT. 0.0 .AND. ABS(TRM) .LT. 1.E-04) TRM2=0.0            00125300
      TRM2=(2.*C1*Q1+C2)*SQRT(TRM2)                                     00125400
      FINT1=(TRM1-TRM2)/(4.*C1)                                         00125500
      FINT2=(4.*C1*C3-C2**2.)/(8.*C1*SQRT(-C1))                         00125600
      TRM3=SQRT(C2**2.-4.*C1*C3)                                        00125700
      ARG1=(2.*C1*Q1+C2)/TRM3                                           00125800
      IF( ABS(ARG1) .GT. 1.) ARG1=SIGN(1.,ARG1)                         00125900
      ARG2=(2.*C1*Q2+C2)/TRM3                                           00126000
      IF( ABS(ARG2) .GT. 1.) ARG2=SIGN(1.,ARG2)                         00126100
      FINT3=ARSIN(ARG1)-ARSIN(ARG2)                                     00126200
      EINTG=(FINT1+FINT2*FINT3)/(XC+1.)                                 00126300
      ELCURV=ABS(EPART1+((-1.)**(WHR2-1.))*EINTG)                       00126400
      IF(     (   (W2 .GT. 0.0) .AND. (AMOD(WHR2,2) .EQ. 1.)        )   00126500
     1                              .OR.                                00126600
     2  (   (W2 .LT. 0.0) .AND. (AMOD(WHR2,2) .EQ. 0.0)      ) ) GO TO 60 00126700
      IF( W2 .NE. 0.0) GO TO 50                                         00126800
      IF(    (   (W1 .GT. 0.0) .AND. (AMOD(WHR1,2) .EQ. 1.))            00126900
     1                              .OR.                                00127000
     2 (   (W1 .LT. 0.0) .AND.(AMOD(WHR1,2) .EQ.0.0))) GO TO 60         00127100
   50 AREA=TCURV-ELCURV                                                 00127200
   55 AREA=AMAX1(0.0,AREA)                                              00127300
      RETURN                                                            00127400
   60 AREA=ELCURV-TCURV                                                 00127500
      GO TO 55                                                          00127600
   75 IF( CORW2 .LT. 0.0) GO TO 76                                      00127700
      G=0.5+CORW2                                                       00127800
      W2=-0.5                                                           00127900
      W1=CORW1-G                                                        00128000
      GO TO 77                                                          00128100
   76 G=0.5+CORW1                                                       00128200
      W1=-0.5                                                           00128300
      W2=CORW2-G                                                        00128400
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```
30    CONTINUE                                                    00128700
      RETURN                                                      00128800
      END                                                         00128900
                                                                  SEGMENT
```

```
                                                                  START OF SEGMENT
      FUNCTION WEIGHT( NPA,NAN)                                   00140100
      COMMON /CBB/ GREGC(14,14)                                   00140200
      IF( NPA .GE. 15) GO TO 10                                   00140300
      WEIGHT=GREGC(NPA,NAN)                                       00140400
      RETURN                                                      00140500
10    IF( NAN .GT. 7) GO TO 20                                    00140600
      WEIGHT=GREGC(14,NAN)                                        00140700
      RETURN                                                      00140800
20    IF(NAN .LE. (NPA-7)) GO TO 30                               00140900
      INDX=14-NPA+NAN                                             00141000
      WEIGHT=GREGC(14,INDX)                                       00141100
      RETURN                                                      00141200
30    WEIGHT=1.                                                   00141300
      RETURN                                                      00141400
      END                                                         00141500
                                                                  SEGMENT
```

```
      FUNCTION  TERP05(UCOR,ZCOR)                                          00129000
      COMMON /CB7/ LSTP,ISUR                                               00129100
      COMMON /CB1/ GRIDW,GRIDQ,GRIDR                                       00129200
      COMMON /CB12/ NUMW,NUMQ,NUMR                                         00129300
      COMMON /CB6/ TMAXW,TMINW,TMAXQ,TMINQ                                 00129400
      COMMON /CB2/ JREC,TOTAT                                              00129500
      COMMON /CB13/ QBEG,WBEG                                              00129600
      COMMON /CB14/ XMEAN(2),XBR1,XBR2,VAR,DGF                             00129700
      COMMON /CB10/EXCES1,EXCES2,EXCES3,EXCES4,A,B,C                       00129800
      COMMON /CB3/ NCONW,NCONQ,NSTRT                                       00129900
      COMMON/CB8/JOINT                                                     00130000
      COMMON /CB20/ RECMAX
      DIMENSION XVAL(10),YVAL(10)                                          00130100
      TERP05=0.0                                                           00130200
      INZER=0                                                              00130300
      IDID=0                                                               00130400
      TERP05=0.0                                                           00130500
      W=A-ZCOR                                                             00130600
      Q=B-UCOR                                                             00130700
      R=C-(UCOR**2.+ZCOR**2.)                                              00130800
      VCH=POSPROB(W,Q,R,LSTP)                                              00130900
      IF( VCH .LT. 0.0) RETURN                                             00131000
      IF( LSTP .GT. ISUR) GO TO 5                                          00131100
      TERP05=CHISQ(VCH,DGF)*PHI(W,XBR1,VAR)*PHI(Q,XBR2,VAR)*               00131200
     1    PHI(UCOR,XMEAN(2),1.)*PHI(ZCOR,XMEAN(1),1.)                      00131300
      RETURN                                                               00131400
5     LB1=(W/GRIDW)+IFIX(NUMW/2.)+1-IFIX((ISUR*XMEAN(1))/GRIDW)            00131500
      WBEG=(LB1-1-IFIX(NUMW/2)+IFIX((ISUR*XMEAN(1))/GRIDW))*GRIDW          00131600
      LB2=(Q/GRIDQ)+IFIX(NUMQ/2)+1-IFIX((ISUR*XMEAN(2))/GRIDQ)             00131700
      QBEG=(LB2-1-IFIX(NUMQ/2)+IFIX((ISUR*XMEAN(2))/GRIDQ))*GRIDQ          00131800
      IF(( W .GT. TMAXW .OR. W .LT. TMINW)      .AND.                      00131900
     1  (Q .GT. TMAXQ .OR. Q .LT. TMINQ)) GO TO 55                         00132000
      IF( WBEG .EQ. W .AND. QBEG .EQ. Q) GO TO 15                          00132100
      IF( WBEG .EQ. W) GO TO 30                                            00132200
      IF( QBEG .EQ. Q) GO TO 40                                            00132300
      GO TO 55                                                             00132400
15    IF( W .GT. TMAXW .OR. W .LT. TMINW) GO TO 40                         00132500
      IF( Q .GT. TMAXQ .OR. Q .LT. TMINQ) GO TO 30                         00132600
      WN=W                                                                 00132700
      QN=Q                                                                 00132800
20    L4=(R/GRIDR)-IFIX((W**2.+Q**2.)/(NSTRT*GRIDR))                       00132900
      IF( L4 .LE. 0 .OR. L4 .GT. NUMR) GO TO 25                            00133000
      CALL RCAL(LB1,LB2,L4,0,WN,QN,RN)                                     00133100
      IF( RN .NE. R) GO TO 25                                              00133200
      IF( JREC .LE. 0.0 .OR. JREC .GT. RECMAX) GO TO 25                    00133250
      READ(JOINT=JREC) PROB3                                               00133300
      TERP05=PROB3*PHI(UCOR,XMEAN(2),1.)*PHI(ZCOR,XMEAN(1),1.)             00133400
      RETURN                                                               00133500
25    TERP05=TERP01(W,Q,R,JOINT)                                           00133600
     1 *PHI(UCOR,XMEAN(2),1.)*PHI(ZCOR,XMEAN(1),1.)                        00133700
      RETURN                                                               00133800
30    LB=LB2                                                               00133900
      IF((LB-1) .LE. 0) LB=2                                               00134000
      IF((LB+1) .GT. NUMQ) LB=NUMQ-1                                       00134100
      L9=LB-1                                                              00134200
      L10=LB+1                                                             00134300
      QPFI=0                                                               00134600
      ZPT=A-W                                                              00134700
      DO 35 L11=L9,L10                                                     00134800
      QH=(L11-1-IFIX(NUMQ/2.)+IFIX((ISUR*XMEAN(2))/GRIDQ))*GRIDQ           00134900
      UPT=B-QH                                                             00135000
      IDID=IDID+1                                                          00135100
                                                                           00135200
```

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      IF( YVAL(IDID) .GT. 0.0) INZER=INZER+1                                00135400
35    CONTINUE                                                              00135500
      CALL BESTINTERP(XVAL,YVAL,IDID,INZER,QPFI,TERPO5)                     00135600
      IF( TERPO5 .LT. 0.0) TERPO5=0.0                                       00135700
      RETURN                                                                00135800
40    M8=LB1                                                                00135900
      IF((M8-1) .LE. 0) M8=2                                                00136000
      IF((M8+1) .GT. NUMW) M8=NUMW-1                                        00136100
      M9=M8-1                                                               00136200
      M10=M8+1                                                              00136300
      WPFI=W                                                                00136600
      QH=Q                                                                  00136700
      QPT=B-QH                                                              00136800
      DO 45 M11=M9,M10                                                      00136900
      WH=(M11-1-IFIX(NUMW/2.))*IFIX((ISUR*XMEAN(1))/GRIDW))*GRIDW           00137000
      ZPT=A-WH                                                              00137100
      IDID=IDID+1                                                           00137200
      XVAL(IDID)=WH                                                         00137300
      YVAL(IDID)=TERPO5(QPT,ZPT)                                            00137400
      IF( YVAL(IDID) .GT. 0.0) INZER=INZER+1                                00137500
45    CONTINUE                                                              00137600
      CALL BESTINTERP(XVAL,YVAL,IDID,INZER,WPFI,TERPO5)                     00137700
      IF( TERPO5 .LT. 0.0) TERPO5=0.0                                       00137800
      RETURN                                                                00137900
50    QPT1=QBEG                                                             00138000
      WPT1=WBEG                                                             00138100
      QPT2=QPT1+GRIDQ                                                       00138200
      WPT2=WPT1+GRIDW                                                       00138300
      QI=Q                                                                  00138400
      WI=W                                                                  00138500
      CALL OVERFL(IND)                                                      00138600
      TLAGR1=(QI-QPT2)*(WI-WPT2)*TERPO5(B-QPT1,A-WPT1)                      00138900
      TLAGR2=(QI-QPT2)*(WI-WPT1)*TERPO5(B-QPT1,A-WPT2)                      00139000
      TLAGR3=(QI-QPT1)*(WI-WPT2)*TERPO5(B-QPT2,A-WPT1)                      00139100
      TLAGR4=(QI-QPT1)*(WI-WPT1)*TERPO5(B-QPT2,A-WPT2)                      00139200
      DVAL=TLAGR1+TLAGR2+TLAGR3+TLAGR4                                      00139300
      IF( DVAL .LT. 0.0 .OR. IND .EQ. 3) GO TO 51                           00139400
      TERPO5=DVAL/(GRIDQ*GRIDW)                                             00139500
51    RETURN                                                                00139600
55    TERPO5=TERPOI(W,Q,R,JOINT)                                            00139700
     1 *PHI(ZCOR,XMEAN(2),1.)*PHI(ZCOR,XMEAN(1),1.)                         00139800
      RETURN                                                                00139900
      END                                                                   00140000
                                                                           SEGMENT
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```
                                                                    START OF SEGMENT
      BLOCK DATA
      COMMON /CB8/GREGC(14,14)
C
C        THESE ARE THE WEIGHTS FOR THE NEWTON-GREGORY INTEGRATION FORMUL
C
      DATA GREGC(1,1)/0.0/
      DATA (GREGC(3,I),I=1,3)/.4166667,1.1166666,.4166667/
      DATA(GREGC(4,I),I=1,4)/.375,1.125,1.125,.375/
      DATA(GREGC(5,I),I=1,5)/.348611,1.2722222,.7583333,1.272222,
     1  .348611/
      DATA(GREGC(6,I),I=1,6)/.3298611,1.3020833,.8680555,.8680555,
     1 1.3020833,.3298611/
      DATA(GREGC(7,I),I=1,7)/.3298611,1.3020833,.7479167,1.2027777
     1,.7479167,1.3020833,.3298611/
      DATA(GREGC(8,I),I=1,8)/.3155919,1.3921792,.6382440,1.1539848,
     11.1539848,.6382440,1.3921792,.3155919/
      DATA(GREGC(9,I),I=1,9)/.3155919,1.3921792,.6239749,1.2583499,
     1.8198082,1.2583499,.6239749,1.3921792,.3155919/
      DATA(GREGC(10,I),I=1,10)/.3155919,1.3921792,.6239749,1.2440807,
     1.9241733,.9241733,1.2440807,.6239749,1.3921792,.3155919/
      DATA(GREGC(11,I),I=1,11)/.3155919,1.3921792,.6239749,1.2440807,
     1.9099041,1.0285384,.9099041,1.2440807,.6239749,1.3921792,.3155919/
      DATA(GREGC(12,I),I=1,12)/.3155919,1.3921792,.6239749,1.2440807,
     1.9099041,1.0142692,1.0142692,.9099041,1.2440807,.6239749,1.3921792
     2,.3155919 /
        DATA(GREGC(13,I),I=1,13)/.3042245,1.4603836,.4534640,1.4714286,
     1.7393932,1.0824735,.9772652,1.0824735,.7393932,1.4714286,.4534640,
     21.4603836,.3042245/
        DATA(GREGC(14,I),I=1,14)/.3042245,1.4603836,.4534640,1.4714286,
     1.7393932,1.0824735,.9886326,.9886326,1.0824735,.7393932,1.4714286,
     3.4534640,1.4603836,.3042245/
      END
                                                                    SEGMENT
```

```
      SUBROUTINE BESTINTERP(X,Y,IDON,INOTZ,XINT,YINT)                    00144700
      DIMENSION X(10),Y(10),EXT(10),WH(10)                               00144800
      YINT=0.0                                                           00144900
C                                                                        00145000
C                                                                        00145100
C        THIS SUBROUTINE DECIDES WHICH TYPE                              00145200
C        OF INTERPOLATION IS MOST APPROPRIATE FOR A PARTICULAR POINT     00145300
C        SINCE THE DENSITY FUNCTION MAY BE TRUNCATED                     00145400
C        LAGRANGIAN INTERPOLATION MAY NOT ALWAYS BE THE BEST             00145500
C                                                                        00145600
C                                                                        00145700
      IF( INOTZ .LE. 0) RETURN                                           00145800
      IF( INOTZ .NE. IDON) GO TO 7                                       00145900
      IF(XINT.LT. X(1).AND. XINT .LT. X(IDON) .AND. Y(1) .LE. 0) RETURN  00146000
      IF(XINT.GT. X(1).AND.XINT.GT.X(IDON).AND.Y(IDON).LE.0.0)RETURN     00146100
      IF(XINT .LT. X(1).AND.XINT.LT.X(IDON)) GO TO 3                     00146200
      IF(XINT .GT. X(1) .AND. XINT .GT. X(IDON)) GO TO 3                 00146300
      DO 2 IRUN=2,IDON                                                   00146400
      IF((XINT.GT. X(IRUN-1).AND. XINT .LT. X(IRUN)) .OR.                00146500
     1(XINT.LT. X(IRUN-1) .AND. XINT .GT. X(IRUN))) INDEX=IRUN-1         00146600
      Y(IRUN)=ALOG(Y(IRUN))                                              00146700
2     CONTINUE                                                           00146800
      Y(1)=ALOG(Y(1))                                                    00146900
      YINT=SPLINEFIT(X,Y,IDON,INDEX,XINT)                                00147000
      IF( YINT .LT. -30.) RETURN                                         00147100
      YINT=EXP(YINT)                                                     00147200
      RETURN                                                             00147300
3     DINT=POLY(X,Y,XINT,IDON)                                           00147400
      IF( DINT .LE. 0.0) RETURN                                          00147500
      YINT=DINT                                                          00147600
      RETURN                                                             00147700
7     DO 8 IRUN=2,IDON                                                   00147800
      IF((XINT .GT. X(IRUN-1) .AND. XINT .LT. X(IRUN)) .OR.              00147900
     1(XINT .LT. X(IRUN-1) .AND. XINT .GT. X(IRUN))) GO TO 9             00148000
      GO TO 8                                                            00148100
9     IF(Y(IRUN-1) .LE. 0.0 .AND. Y(IRUN) .LE. 0.0) RETURN               00148200
      INDEX=IRUN                                                         00148300
      JCT=0                                                              00148400
      IF(Y(IRUN-1) .LE. 0.0 .OR. Y(IRUN) .LE. 0.0) GO TO 25              00148500
      GO TO 10                                                           00148600
8     CONTINUE                                                           00148700
      RETURN                                                             00148800
10    DO 15 JK=1,IDON                                                    00148900
      IF(Y(JK) .LE. 0.0) GO TO 15                                        00149000
      JCT=JCT+1                                                          00149100
      WH(JCT)=X(JK)                                                      00149200
      EXT(JCT)=ALOG(Y(JK))                                               00149300
      IF(JK .EQ. INDEX) IDX=JK                                           00149400
15    CONTINUE                                                           00149500
      IF(JCT .GE. 3) GO TO 20                                            00149600
      DINT=POLY(WH,EXT,XINT,JCT)                                         00149700
      IF( DINT .LE. -35.0) RETURN                                        00149800
      IF( DINT .GT. 0.0) GO TO 3                                         00149900
      YINT=EXP(DINT)                                                     00150000
      RETURN                                                             00150100
20    DINT=SPLINEFIT(WH,EXT,JCT,IDX,XINT)                                00150200
      IF(DINT .LE. -35.) RETURN                                          00150300
      IF( DINT .GT. 0.0) GO TO 3                                         00150400
      YINT=EXP(DINT)                                                     00150500
      RETURN                                                             00150600
25    IF(Y(IRUN-1) .GT. 0.0)FAC=ABS(IFIX(ALOG10(Y(IRUN-1))))             00150700
      IF(Y(IRUN) .GT. 0.0) FAC=ABS(IFIX(ALOG10(Y(IRUN))))                00150800
      DO 50 JK=1,IDON                                                    00150900
      IF( Y(JK) .GT. 0.0) GO TO 30                                       00151000
```

```
      IF( JK .NE. INDEX .OR. JK .NE. (INDEX-1)) GO TO 50                00151000
30    JCT=JCT+1                                                         00151100
      WH(JCT)=X(JK)                                                     00151200
      EXT(JCT)=ALOG(1.+(10.**FAC)*Y(JK))                                00151300
      IF(JK .EQ. INDEX) IDX=JK                                          00151400
50    CONTINUE                                                          00151500
      IF( JCT .GE. 3) GO TO 60                                          00151600
      DINT=POLY(WH,EXT,XINT,JCT)                                        00151700
      GO TO 61                                                          00151800
60    DINT=SPLINE1(WH,EXT,JCT,IDX,XINT)                                 00151900
61    IF(DINT .LT. 0.0) RETURN                                          00152000
      YINT=(EXP(DINT)-1.)/(10.**FAC)                                    00152100
      RETURN                                                            00152200
      END                                                               00152300
                                                                       SEGMENT


      FUNCTION POLY(X,Y,XINT,INUM)                                      00156900
      DIMENSION X(10),Y(10)                                             00157000
C                                                                       00157100
C        THIS ROUTINE PERFORMS INTERPOLATION VIA                        00157200
C        GENERALIZED ONE DIMENSIONAL LAGRANGE                           00157300
C                                                                       00157400
      DSUM=0.0                                                          00157500
      DO 5 IK=1,INUM                                                    00157600
      ITERM=1.                                                          00157700
      BTERM=1.                                                          00157800
      DO 4 JK=1,INUM                                                    00157900
      IF( JK .EQ. IK) GO TO 4                                           00158000
      ITERM=(XINT-X(JK))*ITERM                                          00158100
      BTERM=(X(IK)-X(JK))*BTERM                                         00158200
4     CONTINUE                                                          00158300
      DSUM=DSUM+(ITERM/BTERM)*Y(IK)                                     00158400
5     CONTINUE                                                          00158500
      POLY=DSUM                                                         00158600
      RETURN                                                            00158700
      END                                                               00158800
                                                                       SEGMENT


                                                                  START OF SEGMENT
      FUNCTION ECHECK(INUM,VAL)                                         00158900
      COMMON/ERR/ IRP1,IRP2                                             00159000
      GO TO (10,20                                                ),INUM 00159100
C        THIS IS A CHECK ON ZRANGE- ELLIPSE                             00159200
10    IF(VAL .LT. 0.0 .AND. ABS(VAL) .GT. 1.E-06) IRP1=IRP1+1           00159300
      VAL=AMAX1(0.0,VAL)                                                00159400
      RETURN                                                            00159500
C        THIS IS A CHECK ON ZRANGE - CIRCL                              00159600
20    IF( VAL .LE. 0.0 .AND. ABS(VAL) .GT. 1.E-06) IRP2=IRP2+1          00159700
      RETURN                                                            00159800
      END                                                               00159900
                                                                       SEGMENT
```

```
                                                                    START OF SEGMENT
      FUNCTION SPLINEFIT(X,Y,M,J,XINT)                                      00152400
      DIMENSION X(10),Y(10),D(10),P(10),E(10),C(4,10)                       00152500
      DIMENSION A(10,3),B(10),Z(10)                                         00152600
C                                                                           00152700
C        SPLINEFIT PERFORMS INTERPOLATION BY FITTING                        00152800
C        A CUBIC SPLINE FUNCTION TO THE POINTS                              00152900
C                                                                           00153000
      MM=M-1                                                                00153100
      DO 2 K=1,MM                                                           00153200
      D(K)=X(K+1)-X(K)                                                      00153300
      P(K)=D(K)/6.                                                          00153400
2     E(K)=(Y(K+1)-Y(K))/D(K)                                               00153500
      DO 3 K=2,MM                                                           00153600
3     B(K)=E(K)-E(K-1)                                                      00153700
      A(1,2)=-1.-D(1)/D(2)                                                  00153800
      A(1,3)=D(1)/D(2)                                                      00153900
      A(2,3)=P(2)-P(1)*A(1,3)                                               00154000
      A(2,2)=2.*(P(1)+P(2))-P(1)*A(1,2)                                     00154100
      A(2,3)=A(2,3)/A(2,2)                                                  00154200
      B(2)=B(2)/A(2,2)                                                      00154300
      IF( M .EQ. 3) GO TO 5                                                 00154400
      DO 4 K=3,MM                                                           00154500
      A(K,2)=2.*(P(K-1)+P(K))-P(K-1)*A(K-1,3)                               00154600
      B(K)=B(K)-P(K-1)*B(K-1)                                               00154700
      A(K,3)=P(K)/A(K,2)                                                    00154800
4     B(K)=B(K)/A(K,2)                                                      00154900
5     Q=D(M-2)/D(M-1)                                                       00155000
      A(M,1)=1.+Q+A(M-2,3)                                                  00155100
      A(M,2)=-Q-A(M,1)*B(M-1)                                               00155200
      Z(M)=B(M)/A(M,2)                                                      00155300
      MN=M-2                                                                00155400
      DO 6 I=1,MN                                                           00155500
      K=M-I                                                                 00155600
6     Z(K)=B(K)-A(K,3)*Z(K+1)                                               00155700
      Z(1)=-A(1,2)*Z(2)-A(1,3)*Z(3)                                         00155800
      DO 7 K=1,MM                                                           00155900
      Q=1./(6.*D(K))                                                        00155950
      C(1,K)=Z(K)*Q                                                         00156000
      C(2,K)=Z(K+1)*Q                                                       00156100
      C(3,K)=Y(K)/D(K)-Z(K+1)*P(K)                                          00156200
7     C(4,K)=Y(K+1)/D(K)-Z(K+1)*P(K)                                        00156300
      TINT=(X(J+1)-XINT)*(C(1,J)*(X(K+1)-XINT)**2.+C(3,J))                  00156400
      TINT=TINT+ (XINT-X(J))*(C(2,J)*(XINT-X(J))**2.+C(4,J))                00156500
      SPLINEFIT=TINT                                                        00156600
      RETURN                                                                00156700
      END                                                                   00156800
                                                                    SEGMENT
```

```
      SUBROUTINE RESUME(X1,X2,X3,*)
C
C
C          THIS SUBROUTINE ALLOWS THE PROGRAM TO BE
C          RUN FOR A PERIOD OF TIME AND TERMINATED
C          BY THE COMPUTER OPERATOR. THIS SUB INITIALIZES
C          ALL THE IMPORTANT VARIABLES BACK TO WHAT THEY WERE WHEN THE
C          PROGRAM WAS RUNNING
C
C
      COMMON/CB7/ LSTP,ISUR
      COMMON /CB10/ CA,PA,CR,PR,A,B,C
      COMMON /CB8/ JOINT,ICAL
      COMMON/RESTAR/OC(30,2),ASN(30),NTESTS
      COMMON /RESTAR/ KTEST,NOC,NSTP,I1,I2,I3,KREC,IRAC,NPFIAC,IRNR
      COMMON /RESTAR/ NPFINR,PROBAC,PROBNR,PRRAC,PRQAC,PRRNR,PRQNR
      COMMON /RESTAR/ RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR
      COMMON /RESTAR/ WN,QN,RN
      DAY="  /  /"
      YEAR="  "
      DYM=TIME(5)
      DOFWK=TIME(6)
      TOD=TIME(1)/216000.0
      DAY=CONCAT(DAY,DYM,12,12,12)
      DAY=CONCAT(DAY,DYM,30,24,12)
      YEAR=CONCAT(YEAR,DYM,12,36,12)
      READ(10=1)   NTRYS,KTEST,NOC,NSTP,JOINT,ICAL,I1,I2,I3,KREC,IRAC,
     1 NPFIAC,IRNR,NPFINR,A,B,C,PROBAC,PROBNR,PRRAC,PRQAC,PRRNR,PRQNR,
     2  RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR,X1,X2,X3
      NTRYS=NTRYS+1
      WRITE(8,100) NTRYS,DOFWK,DAY,YEAR,TOD
100   FORMAT(1H1,20X,"RESUMING PROCESSING",20X,"TRY",I5,//,10X,A6,10X,
     1 2A6,25X,"AT",E15.7,"   HOURS")
      WRITE(8,101)
101   FORMAT(    40X,"SUMMARY FROM LAST RUN")
      WRITE(8,*)  NTRYS,KTEST,NOC,NSTP,JOINT,ICAL,I1,I2,I3,KREC,IRAC,
     1 NPFIAC,IRNR,NPFINR,A,B,C,PROBAC,PROBNR,PRRAC,PRQAC,PRRNR,PRQNR,
     2  RVALAC,RVALNR,RACBEG,SPROAC,RNBEG,SPRONR,X1,X2,X3
                                                                  SEGMENT
      WRITE(8,102)
102   FORMAT(///,40X,"SUMMARY FROM ALL PREVIOUS RUNS")
      DO 10 J1=1,KTEST
      DO 10 J2=1,NOC,9
      DO 10 J3=1,NSTP-1
      READ(11=(J3+((J2-1)*NTESTS)+(J1-1)*NTESTS*10)) PACC,PREJ,PCON,
     1    NSTEP,ALAM,NCASE
      WRITE(8,*) NCASE,ALAM,NSTEP,PACC,PREJ,PCON
      IF( NSTP .LE. (ISUR+1)) GO TO 10
      IF( J1 .LT. KTEST .OR. J2 .LT. NOC .OR. J3 .LE. ISUR) GO TO 10
      OC(J3,1)=PACC
      OC(J3,2)=PREJ
      ASN(J3)=PCON
10    CONTINUE
      WN=A
      QN=B
      RN=C
      RETURN 1
      END
```

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The Sequential Analysis of Variance test (SANOVA) is introduced in chapter one. The thesis will show how to obtain the OC and and ASN of such a test. Currently only an approximation exists due to Bhate (1959) discussed in chapter one. In chapter two the first exact procedure for obtaining the OC and ASN of a k=2 SANOVA test is derived. Section (2.3) provides the theoretical procedure, summarized in Figures 12 and 13 of section (2.5).

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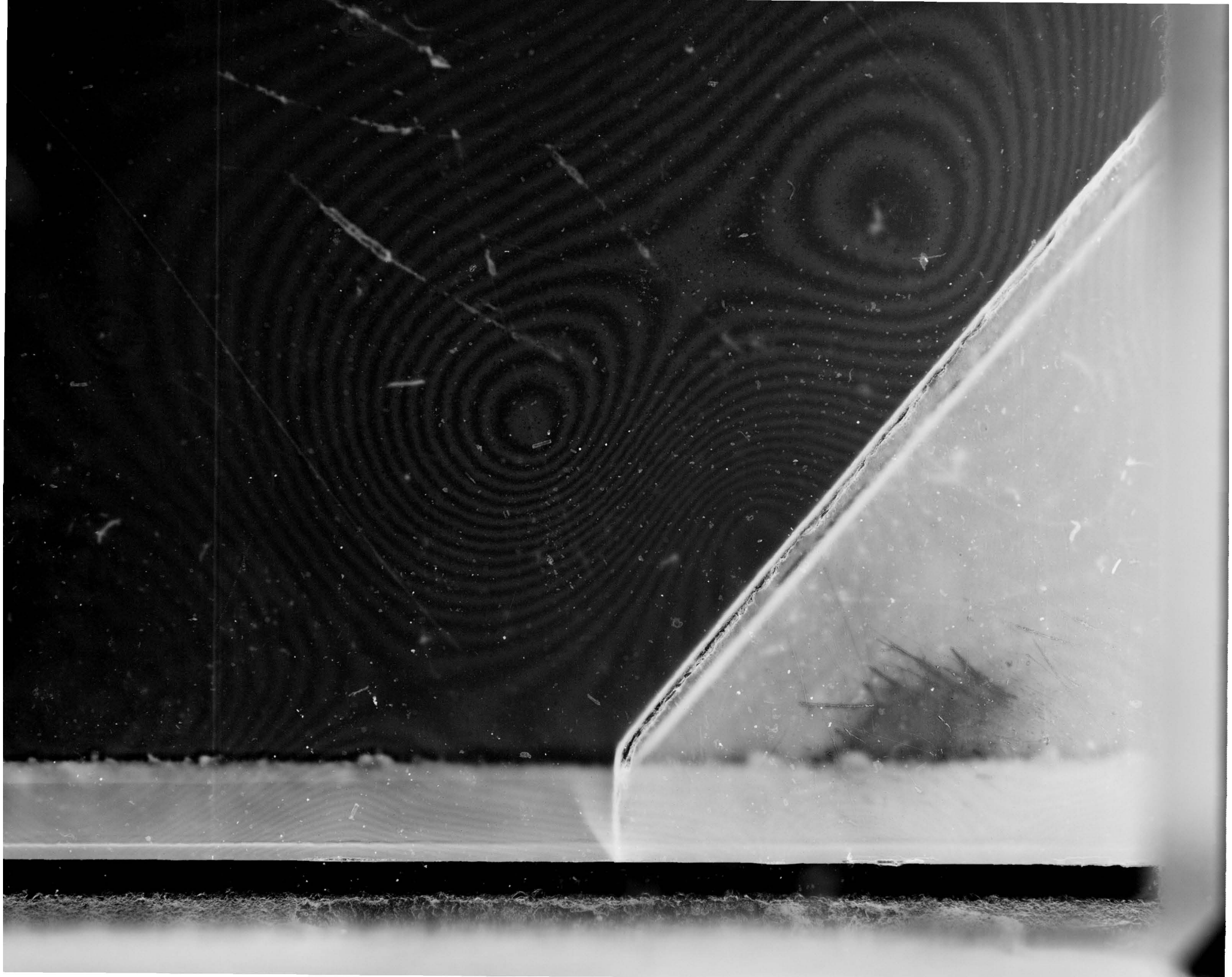
The numerical approach is discussed ~~in section (2.6)~~. Appendix A gives the power calculation for a fixed sample ANOVA test; Appendix B shows how the Wald regions are found; Appendix C contains a computer program for the OC and ASN.